



# On the Analysis of Damped Gyroscopic Systems Using Lyapunov Direct Method

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## Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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## Abstract

In this work, the stability properties of damped gyroscopic systems have been studied using Lyapunov direct method. These systems are generally stable because of the presence of gyroscopic effect. Conditions for determining the stability of the damped gyroscopic systems have been developed. Solution bounds of amplitude and velocity have been obtained for both homogeneous and inhomogeneous cases. An example is given to show how the stability conditions are applied to systems to determine its stability status.

*Keywords:* Asymptotic stability; damped gyroscopic systems; stability; solution bounds.

## 1 Introduction

A matrix system with mass, gyroscopic, stiffness matrices and excitation is said to be a gyroscopic system. Gyroscopic systems find wide usage in engineering applications. They can be major system component themselves or sub-components of larger or more complicated systems. Common examples include rotating shafts when treated in a rotating co-ordinate frame, pipes conveying fluid and elastic strips moving in an axial direction. When modeling their transverse vibrations, each of these simple systems is formulated as a gyroscopic

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system. Such systems are widely known to exhibit interesting stability property whose analysis is non-trivial. Recently, studies have been made on the stability of gyroscopic systems [1-12].

The addition of damping matrix to a gyroscopic system gives a damped gyroscopic system. Applications abound in vibrating systems such as beams, building, bridges, highways, large space structures etc. If the gyroscopic force is not present, the system is called non-gyroscopic system. The relationship between stability, damping and gyroscopic forces is described by the Kelvin-Tait-Chetaev theorem. If a system is stable in the absence of gyroscopic and dissipative forces, their addition to the system will lead to asymptotic stability. Conversely, if an unstable equilibrium can be stabilized by the addition of pure gyroscopic forces, then, depending on whether or not it exhibits complete dissipation, the addition of damping may enhance or destroy this stability. Depending on the coefficients of a particular gyroscopic system, internal damping forces may give rise to complete dissipation.

The damped gyroscopic systems are generally stable systems. Due to this and their common usage in industrial applications, where there is a desire to increase mechanical efficiency and operational safety and to minimize noise and vibration, further study of the stability of this class of system would be beneficial. In this work, damped gyroscopic system is studied using Lyapunov direct method. Stability conditions for determining the stability or otherwise of the system are provided. Example is given to illustrate the efficacy of the result.

## 2 Preliminaries

Consider the homogeneous linear damped gyroscopic system

$$M\ddot{x} + (D + G)\dot{x} + Kx = 0 \quad (1)$$

where  $M$ ,  $D$  and  $K$  which are Hermitian and positive definite are the mass matrix, the damping matrix and the stiffness matrix respectively. The matrix  $G$  of the gyroscopic forces is skew-Hermitian.

The solution of (1) is assumed to be of the form

$$x = qe^{\lambda t} \quad (2)$$

(where  $q$  is an arbitrary constant and  $\lambda$  is an eigenvalue).

Substituting (2) in (1) we have the following

$$e^{\lambda t} (\lambda^2 M + \lambda(D + G) + K)q = 0 \quad (3)$$

where  $e^{\lambda t} \neq 0$  and  $q \neq 0$ .

The eigenvalues of (3) can be used to investigate the stability of system (1).

The eigenvalues  $\lambda$  are the solutions of the characteristics polynomial

$\det (\lambda^2 M + \lambda(D + G) + K)$  of degree  $2n$ .

### **Routh-Hurwitz Criterion**

For the roots of the polynomial

$$a_1 \lambda^n + a_2 \lambda^{n-1} + a_3 \lambda^{n-2} + \dots + a_n = 0$$

to have negative real part it is necessary that

$$\frac{a_2}{a_1} > 0, \frac{a_3}{a_1} > 0 \dots \frac{a_n}{a_1} > 0$$

Using the above criterion shows that the system is stable and if all the eigenvalues have negative real parts, then, the system is said to be asymptotically stable. The stability of system (1) can also be investigated using Thomson-Tait-Cetaev theorem [13]. Here we discuss the stability of the system by the direct method of Lyapunov.

The system (1) is equivalent to the system

$$\left. \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -M^{-1}(D+G)x_2 - M^{-1}Kx_1 \end{aligned} \right\} \quad (4)$$

The equivalent form of a first order system of (3) is given as

$$\dot{z} = Az \quad (5)$$

where

$$\dot{z} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \dot{x}_1 \\ \ddot{x}_1 \end{pmatrix}, \quad z = \begin{pmatrix} x_1 \\ \dot{x}_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$A = \begin{bmatrix} O & I \\ -M^{-1}K & -M^{-1}(D+G) \end{bmatrix}$$

where I is the identity matrix and O is the zero matrix.

Let  $V(z(t))$  be a Lyapunov function for system (5). If  $V > 0$  and the time derivative  $\dot{V} \leq 0$  for all solutions  $z(t)$  of (5), the existence of such a Lyapunov function implies stability of the system (asymptotic stability if  $\dot{V} < 0$ ). Since Lyapunov functions are considered generalized energy expressions, it is proper to consider  $V$  as a quadratic form in the co-ordinates and in the velocities.

$$V = z(t)^* P z(t)$$

with a Hermitian matrix  $P > 0$ . For the solutions of (5) we then have

$$V = z(t)^* (A^* P + PA) z(t), \text{ such that condition } \dot{V} \leq 0 \text{ is expressed by the matrix } Q = Q^* \geq 0 \text{ of the}$$

Lyapunov matrix equation.

$$A^* P + PA = -Q$$

The stability or asymptotic stability is assured if there exists Hermitian matrices  $P > 0$  and  $Q > 0$  which satisfy the Lyapunov matrix equation.

## 2.1 Derivation of P and Q

We now derive suitable positive Hermitian matrices P and Q satisfying the Lyapunov equation. This is obtained from the Lyapunov function. To obtain the Lyapunov function, we start with the energy equation. Multiplying (1) from left with  $\dot{x}^*(t)$  we have the following

$$\dot{x}^*M\dot{x} + \dot{x}^*(D+G)\dot{x} + \dot{x}^*Kx = 0 \quad (6)$$

Adding the complex transpose of (6) to (6) we have

$$2\dot{x}^*M\dot{x} + 2\dot{x}^*D\dot{x} + 2\dot{x}^*Kx = 0 \quad (7)$$

Integrating (7) we have

$$2\dot{x}^*M\dot{x} + x^*Kx + \int_0^t \dot{x}^*D\dot{x}dt = 2\epsilon_0 \quad (8)$$

Where  $2\epsilon_0$  is the integration constant.

Similarly, multiplying system (1) from left by  $x^*$  we have

$$x^*M\ddot{x} + x^*(D+G)\dot{x} + x^*Kx = 0 \quad (9)$$

Adding the conjugate transpose of (9) to (9) we have

$$x^*M\ddot{x} + \dot{x}^*M\dot{x} + 2x^*D\dot{x} + 2x^*K\dot{x} + (x^*G\dot{x} - \dot{x}^*Gx) = 0 \quad (10)$$

Integrating (10) we have

$$\dot{x}^*Mx + x^*M\dot{x} + x^*Dx + \int_0^t (2x^*Kx - 2\dot{x}^*M\dot{x} + (x^*G\dot{x} - \dot{x}^*Gx))dt = c \quad (11)$$

where  $c$  is an integration constant.

In order to obtain the Lyapunov function, we introduce a proper positive constant  $\gamma$ . Multiplying (11) by  $\frac{\gamma}{2}$  we have

$$\frac{\gamma}{2}(\dot{x}^*Mx + x^*M\dot{x} + x^*Dx) + \frac{\gamma}{2}\int_0^t (2x^*Kx - 2\dot{x}^*M\dot{x} + (x^*G\dot{x} - \dot{x}^*Gx))dt = \frac{\gamma}{2}c \quad (12)$$

Adding (8) and (12) we have the following

$$\begin{aligned} & x^*(K + \frac{\gamma}{2}D)x + \dot{x}^*M\dot{x} + \dot{x}^*\frac{\gamma}{2}Mx + x^*\frac{\gamma}{2}M\dot{x} \\ & = 2\epsilon_0 + \frac{\gamma}{2}c - \int_0^t (x^*\gamma Kx + x^*\frac{\gamma}{2}G\dot{x} + \dot{x}^*\left(-\frac{\gamma}{2}G\right)x + \dot{x}^*(2D-\gamma M)\dot{x})dt \end{aligned} \quad (13)$$

Putting (13) in the quadratic form  $V = z^*Pz$

$$\text{where } z^* = \begin{bmatrix} x^* \\ \dot{x}^* \end{bmatrix} \text{ and } z = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

we have

$$V = \begin{bmatrix} x^*P_{11}x & x^*P_{12}\dot{x} \\ \dot{x}^*P_{21}x & \dot{x}^*P_{22}\dot{x} \end{bmatrix}$$

But eqn(13) is in the form

$$V = P + \int_0^t Q(s)ds$$

where using (13) we have

$$P = \begin{bmatrix} K + \frac{\gamma}{2}D & \frac{\gamma}{2}M \\ \frac{\gamma}{2}M & M \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} \gamma K & \frac{\gamma}{2}G \\ \frac{\gamma}{2}G^* & 2D - \gamma M \end{bmatrix}$$

## 2.2 Verification

$$\begin{aligned} A &= \begin{bmatrix} O & I \\ -M^{-1}K & -M^{-1}(D+G) \end{bmatrix} \\ A^* &= \begin{bmatrix} O & -M^{-1}K \\ I & -M^{-1}(D-G) \end{bmatrix} \\ A^*P &= \begin{bmatrix} O & -M^{-1}K \\ I & -M^{-1}(D-G) \end{bmatrix} \begin{bmatrix} K + \frac{\gamma}{2}D & \frac{\gamma}{2}M \\ \frac{\gamma}{2}M & M \end{bmatrix} \\ &= \begin{bmatrix} -\frac{\gamma}{2}K & -K \\ K + \frac{\gamma}{2}G & \frac{\gamma}{2}M - D + G \end{bmatrix} \\ PA &= \begin{bmatrix} -\frac{\gamma}{2}K & K - \frac{\gamma}{2}G \\ -K & \frac{\gamma}{2}M - (D+G) \end{bmatrix} \\ A^*P + PA &= - \begin{bmatrix} \gamma K & \frac{\gamma}{2}G \\ \frac{\gamma}{2}G^* & 2D - \gamma M \end{bmatrix} = -Q \end{aligned}$$

## 3 Stability Analysis

The stability or asymptotic stability of (1) is ensured. We now give the conditions for the existence of a positive definite P and Q that will satisfy the Lyapunov matrix equation ensuring the stability of the damped gyroscopic system.

**Schur's Lemma**

A matrix  $R = \begin{bmatrix} R_1 & R_2 \\ R_2^* & R_3 \end{bmatrix}$  with Hermitian submatrices  $R_1$  and  $R_3$  is positive definite if and only if  $R_1$  and

$R_3 - R_2^* R_1^{-1} R_2$  are positive definite.

The Schur's lemma gives the conditions for the positive definiteness of P and Q.

Applying the lemma to Q, we have that  $Q > 0$  if and only if there exist  $\gamma > 0$  such that

$$\begin{aligned} 2D - \gamma M - \frac{\gamma}{2} G^* (\gamma K)^{-1} \frac{\gamma}{2} G &> 0 \\ -\gamma^2 \left( M + \frac{1}{4} G^* K^{-1} G \right) + \gamma 2D &> 0 \end{aligned} \quad (14)$$

Consider  $z \in \mathbb{C}^n$  and taking  $z^* z = 1$ , then, (14) is equivalent to the inequality

$$-\gamma^2 z^* \left( M + \frac{1}{4} G^* K^{-1} G \right) z + \gamma z^* 2D z > 0 \quad (15)$$

The coefficients of the quadratic polynomial in  $\gamma$  are Rayleigh quotients for Hermitian matrices.

These Rayleigh quotients are limited by the smallest eigenvalue  $\lambda_{\min}$  and the largest eigenvalue  $\lambda_{\max}$  of the respective matrices. Assume  $M, D$  and  $K^{-1}$  to be positive definite, then, the Rayleigh quotients for the matrices  $M, \frac{1}{4} G^* K^{-1} G$  and  $2D$  are also positive definite.

We now introduce the scalars  $a$  and  $b$  defined by

$$\begin{aligned} a &= \lambda_{\max} \left( M + \frac{1}{4} G^* K^{-1} G \right) > 0 \\ b &= \lambda_{\min} 2D > 0 \end{aligned} \quad (16)$$

Let  $\gamma > 0$ , using (16) and (15) we have

$$\begin{aligned} -\gamma^2 a + \gamma b &> 0 \\ \gamma(b - a\gamma) &> 0 \end{aligned}$$

Thus  $\gamma > 0$

and  $b - a\gamma > 0 \Rightarrow \gamma < \frac{b}{a}$

There are solutions  $\gamma > 0$  if and only if

$$b > 0 \text{ and } \frac{b^2}{4a} > 0$$

**Stability Theorem**

Let  $a$  and  $b$  be defined by (16). If  $b > 0$  and  $\frac{b^2}{4a} > 0$  then the system (5) is asymptotically stable.

The constant  $a$  and  $b$  need to be determined. Therefore, it is practical to make the following estimates of  $a$  and  $b$

$$\begin{aligned} a &\leq \lambda_{\max}(M) + \frac{1}{4} \lambda_{\max}(G^* K^{-1} G) \\ b &\geq 2\lambda_{\min}(D) \end{aligned}$$

according to [14], we can further estimate as follows

$$\begin{aligned}\lambda_{\max}(M) &\leq m_{\max} \\ \lambda_{\max}(G^*K^{-1}G) &\leq \frac{g_{\max}^2}{k_{\min}}\end{aligned}\quad (17)$$

$$\lambda_{\min}(D) \leq d_{\min}$$

where  $g_{\max}$  and  $m_{\max}$  are the maximum of the absolute values of the eigenvalues of G and M respectively. And  $k_{\min}$  and  $d_{\min}$  are the smallest eigenvalues of K and D respectively. With the application of (16) and (17), the conditions for the existence of  $\gamma > 0$  become

$$2d_{\min} > 0, \text{ and } \frac{4k_{\min}d_{\min}^2}{4m_{\max}k_{\min} + g_{\max}^2} > 0$$

We can now choose appropriate  $\gamma > 0$  as

$$\begin{aligned}\gamma &= \frac{b}{a} \\ \Rightarrow \gamma &= 8d_{\min}k_{\min}/(m_{\max}k_{\min} + g_{\max}^2)\end{aligned}\quad (18)$$

### 3.1 Solution bounds for the homogeneous case

The stability of the homogeneous system (1) can be established by Thomson-Tait-Cetaev theorem or Routh criterion. Therefore it is assumed to be stable. This is necessary since bounds are not obtained for unstable systems. The stability implies there exists a value  $\gamma > 0$  and a Lyapunov function V for a given solution  $x(t)$ . The Lyapunov function V is given as

$$\begin{aligned}V &= z(t)^* Pz(t) \\ &= x^*(t) \left( K + \frac{\gamma}{2} D - \frac{\gamma^2}{4} M \right) x(t) + \left( \dot{x}(t) + \frac{\gamma}{2} x(t) \right)^* M \left( \dot{x}(t) + \frac{\gamma}{2} x(t) \right) \leq V_0\end{aligned}\quad (19)$$

where  $V_0$  is the initial energy given by the initial condition

$$V_0 = x^*(0) \left( K + \frac{\gamma}{2} D - \frac{\gamma^2}{4} M \right) x(0) + \left( \dot{x}(0) + \frac{\gamma}{2} x(0) \right)^* M \left( \dot{x}(0) + \frac{\gamma}{2} x(0) \right)$$

The energy equation can now be used to establish the solution bounds for the amplitude and velocity. The amplitude bound of  $x(t)$  is obtained by estimating the first term of V as follows

$$0 \leq \lambda_{\min} \left( K + \frac{\gamma}{2} D - \frac{\gamma^2}{4} M \right) x^*(t)x(t) \leq x^*(t) \left( K + \frac{\gamma}{2} D - \frac{\gamma^2}{4} M \right) x(t)\quad (20)$$

From (19) we have that

$$x^*(t) \left( K + \frac{\gamma}{2} D - \frac{\gamma^2}{4} M \right) x(t) \leq V_0\quad (21)$$

Therefore,

$$\lambda_{\min} \left( K + \frac{\gamma}{2} D - \frac{\gamma^2}{4} M \right) x^*(t)x(t) \leq V_0 \quad (22)$$

Introducing the 2-norm  $x^*(t)x(t) = \|x(t)\|^2$  and neglecting the second term of V we have that

$$\begin{aligned} \lambda_{\min} \left( K + \frac{\gamma}{2} D - \frac{\gamma^2}{4} M \right) \|x(t)\|^2 &\leq V_0 \\ \|x(t)\| &\leq \sqrt{\frac{V_0}{\lambda_{\min} \left( K - \frac{\gamma}{2} D - \frac{\gamma^2}{4} M \right)}} \end{aligned} \quad (23)$$

$\|x(t)\|$  in (23) is estimated in terms of  $\gamma$ . It is therefore necessary to estimate  $\gamma$ . We choose

$\gamma = \frac{b}{a}$ . This choice gives the tightest bound for this case. We now find the velocity  $\dot{x}(t)$  in terms of  $\|x(t)\|$ . To obtain  $\|\dot{x}\|$  we estimate the second term of V as follows.

$$\begin{aligned} \left( \dot{x}(t) + \frac{\gamma}{2} x(t) \right)^* M \left( \dot{x}(t) + \frac{\gamma}{2} x(t) \right) &\geq \lambda_{\min} (M) \left( \dot{x}(t) + \frac{\gamma}{2} x(t) \right)^* \left( \dot{x}(t) + \frac{\gamma}{2} x(t) \right) \\ &\geq \lambda_{\min} (M) \left\| \dot{x}(t) + \frac{\gamma}{2} x(t) \right\|^2 \geq \lambda_{\min} (M) \left( \|\dot{x}(t)\| + \frac{\gamma}{2} \|x(t)\| \right)^2 \\ &\geq \lambda_{\min} (M) \left( \|\dot{x}(t)\| - \frac{\gamma}{2} \|x(t)\| \right)^2 \end{aligned} \quad (24)$$

From (19) we have the following

$$\left( \dot{x}(t) + \frac{\gamma}{2} x(t) \right)^* M \left( \dot{x}(t) + \frac{\gamma}{2} x(t) \right) \leq V_0$$

Therefore it implies from (24) that

$$\begin{aligned} \lambda_{\min} (M) \left( \|\dot{x}(t)\| - \frac{\gamma}{2} \|x(t)\| \right)^2 &\leq V_0 \\ \|\dot{x}(t)\| &\leq \frac{\gamma}{2} \|x(t)\| + \sqrt{\frac{V_0}{\lambda_{\min} (M)}} \end{aligned} \quad (25)$$

Using the estimate of  $\|x(t)\|$  in (23), we have the following

$$\|\dot{x}(t)\| \leq \frac{\gamma}{2} \sqrt{\frac{V_0}{\lambda_{\min} \left( K - \frac{\gamma}{2} D - \frac{\gamma^2}{4} M \right)}} + \sqrt{\frac{V_0}{\lambda_{\min} (M)}} \quad (26)$$



It is possible to obtain bounds for every individual co-ordinates. We note that for a given quadratic form  $V = z(t)^* Pz(t)$ ,  $P > 0$ , then for a fixed value  $V$  the upper bound for the co-ordinate  $z_k$  is as follows

$$|z_k| \leq \sqrt{VP_{kk}^{-1}} \quad (27)$$

where  $P_{kk}^{-1}$  is the  $k$ th diagonal element of the inverse matrix  $P^{-1}$ . The individual amplitude bound corresponding to (23) is the following

$$|x_k(t)| \leq \sqrt{V_0 \left( K + \frac{\gamma}{2} D - \frac{\gamma^2}{4} M \right)_{kk}^{-1}} \quad (28)$$

where  $\left( K + \frac{\gamma}{2} D - \frac{\gamma^2}{4} M \right)_{kk}^{-1}$  is the  $k$ th diagonal element of the inverse matrix

$$\left( K + \frac{\gamma}{2} D - \frac{\gamma^2}{4} M \right)^{-1}$$

Similarly, the individual velocity bound corresponding to (25) is obtained as follows

$$\left| \dot{x}_k(t) + \frac{\gamma}{2} x_k(t) \right| \leq \sqrt{V_0 M_{kk}^{-1}} \quad (29)$$

where  $M_{kk}^{-1}$  is the  $k$ th diagonal element of the inverse matrix  $M^{-1}$ .

It follows from (29) that

$$|\dot{x}_k(t) + \frac{\gamma}{2} x_k(t)| \leq \sqrt{V_0 M_{kk}^{-1}}$$

But

$$\begin{aligned} |\dot{x}_k(t) - \frac{\gamma}{2} x_k(t)| &\leq |\dot{x}_k(t) + \frac{\gamma}{2} x_k(t)| \leq \sqrt{V_0 M_{kk}^{-1}} \\ \Rightarrow |\dot{x}_k(t) - \frac{\gamma}{2} x_k(t)| &\leq \sqrt{V_0 M_{kk}^{-1}} \end{aligned}$$

Therefore,

$$|\dot{x}_k(t)| \leq \frac{\gamma}{2} |x_k(t)| + \sqrt{V_0 M_{kk}^{-1}} \quad (30)$$

### 3.2 Solution bounds for the inhomogeneous case

The addition of excitation  $f(t)$  to the homogeneous system(1) gives the inhomogeneous damped gyroscopic system

$$M\ddot{x} + (D + G)\dot{x} + Kx = f(t) \quad (31)$$

The stability of the inhomogeneous system (31) follows from the stability of the homogeneous system(1). The response bounds for the solution  $x(t)$  of the in homogeneous system is therefore obtained in terms of the bounds

of the homogeneous system. For a non-transient excitation  $f(t)$ , the solution  $x(t)$  is made up of the general solution (homogeneous solution) and the particular solution. The solution of system (31) is given as

$$\begin{aligned} x(t) &= x_{\square}(t) + x_{part}(t) \\ \Rightarrow x_h(t) &= x(t) - x_{part}(t) \end{aligned}$$

with the initial conditions for its corresponding state and velocity bounds as  $x_h(0) = x(0) - x_{part}(0)$  and  $\dot{x}_h(0) = \dot{x}(0) - \dot{x}_{part}(0)$ . The initial energy  $V_{0,\square}$  for  $x_{\square}(t)$  is the following

$$V_{0,h} = x_h^*(0) \left( K + \frac{\gamma}{2} D - \frac{\gamma^2}{4} M \right) x_h(0) + \left( \dot{x}_h(0) + \frac{\gamma}{2} x_h(0) \right)^* M \left( \dot{x}_h(0) + \frac{\gamma}{2} x_h(0) \right)$$

Using (23) and (28) we have the following for the amplitude bounds of 2-norm and individual co-ordinate of the inhomogeneous system

$$\|x(t)\| \leq \sqrt{\frac{V_{0,h}}{\lambda_{\min} \left( K + \frac{\gamma}{2} D - \frac{\gamma^2}{4} M \right)}} + \|x_{part}(t)\| \quad (32)$$

and

$$|x_k(t)| \leq \sqrt{V_{0,h} \left( K + \frac{\gamma}{2} D - \frac{\gamma^2}{4} M \right)_{kk}^{-1}} + |x_{part,k}(t)| \quad (33)$$

The velocity bounds for 2-norm and individual co-ordinate of the inhomogeneous system can similarly be obtained respectively in terms of the results of bounds of the homogeneous system as

$$\|\dot{x}(t)\| \leq \frac{\gamma}{2} \|x(t)\| + \sqrt{\frac{V_{0,h}}{\lambda_{\min}(M)}} + \|\dot{x}_{part}(t)\| \quad (34)$$

and

$$|\dot{x}_k(t)| \leq \sqrt{V_{0,\square} \left( K + \frac{\gamma}{2} D - \frac{\gamma^2}{4} M \right)_{kk}^{-1}} + |\dot{x}_{part,k}(t)| \quad (35)$$

For a transient excitation  $f(t)$  we can find a solution to (31) with the initial conditions  $x(0) = 0$  and  $\dot{x}(0) = 0$  by calculating the convolution of the impulse response matrix  $\phi(t)$  and  $f(t)$ . The solution of (31) is as follows:

$$x(t) = \int_0^t \phi(t-\tau) f(\tau) d\tau$$

The impulse response matrix  $\phi(t)$  satisfies

$$M\ddot{\phi} + (D+G)\dot{\phi} + K\phi = 0, \quad \phi(0) = 0, \quad M\dot{\phi}(0) = I$$

where  $I$  is the identity matrix.

We now assume that the excitation vector  $f(t)$  has the form

$$F(t) = u\psi(t)$$

where  $u$  is a constant vector and  $\psi(t)$  is a scalar function subjected to

$$P = \int_0^{\infty} |\psi(t)| dt < \alpha$$

To obtain bounds of solution  $x(t)$  given by (26) we have to estimate the solution to the homogeneous equation  $\dot{\varphi}(t) = \dot{\phi}(t)u$  which satisfies the initial conditions  $\varphi(0) = 0$  and  $\dot{\varphi}(0) = \dot{\phi}(0)u = M^{-1}u$ , and therefore  $V_{0,h}u^* M^{-1}u$ . This leads to the following estimate of the 2-norm of the solution  $x(t)$ .

$$\|x(t)\| \leq \sqrt{\frac{u^* M^{-1}u}{\lambda_{\min}\left(K + \frac{\gamma}{2}D - \frac{\gamma^2}{4}M\right)}} P \quad (36)$$

By using (22), we can also obtain an estimate for the co-ordinate  $x_k(t)$  of the solution  $x(t)$ .

$$|x_k(t)| \leq \sqrt{u^* M^{-1}u \left(K + \frac{\gamma}{2}D - \frac{\gamma^2}{4}M\right)_{kk}^{-1}} P \quad (37)$$

## 4 Application

Consider the damped gyroscopic system

$$M\ddot{x} + (D + G)\dot{x} + Kx = 0$$

where

$$M = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{pmatrix}, \quad D = \begin{pmatrix} 8 & -2 & 2 \\ -2 & 8 & -2 \\ 2 & -2 & 8 \end{pmatrix}$$

$$G = \begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 2 \\ -3 & -2 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 4 & 2 & 3 \\ 2 & 4 & 2 \\ 3 & 2 & 4 \end{pmatrix}$$

To apply the stability theorem, we compute the constants in the theorem as follows

$$a = \lambda_{\max}(M + G^*K^{-1}G) = 8.2$$

$$b = \lambda_{\max}(2D) = 1$$

Since  $b = 1 > 0$  and  $\frac{b^2}{4a} = 0.031 > 0$ , the system is stable according to the stability theorem.

## 5 Conclusion

The stability properties of damped gyroscopic systems have been studied. These systems are generally stable because of the presence of gyroscopic effect. Gyroscopic forces can stabilize unstable systems. The stability of MDK system is not always guaranteed since the gyroscopic effect is absent. Conditions for determining the stability of the damped gyroscopic systems have been developed. Solution bounds are obtained only for stable systems. Solution bounds of amplitude and velocity have been obtained for both homogeneous and inhomogeneous cases. An example is given to show how the stability conditions are applied to systems to determine its stability status.

## Competing Interests

Author has declared that no competing interests exist.

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