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Contribution to the Numerical Solution of Nonlinear Heat Transfer Equation Subject to a Boundary Integral Specification

Djibet Mbainguesse [a*], Bakari Abbo [a] and Youssouf Pare [b]

[a] *Department of Mathematics, University of N'djamena, Chad.*

[b] *Department of Mathematics, University Joseph Ki Zerbo, Burkina Faso.*

Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

The work presents one dimensional heat transfer in a media with temperature-dependent thermal conductivity. We solve numerically the one-dimensional unsteady heat conduction equation subject to initial condition and integral boundary conditions. We first discertize the equation in time, using the implicite Euler time method. A sequence of nonlinear two-point boundary value problems is obtained. This discretisation reduce the problem to the second spatial derivative of temperature wich is a nonlinear function of the temperature and the temperature gradient. For the implementation of Newton method, we derive expressions for the

**Corresponding author: E-mail: mbainguesse@gmail.com;*

partial derivative of the nonlinear function. Using higher order parallel splitting finite difference method and the Simpson's composite quadrature method, we solve the the resulting nonlinear systems by the multivariate Newton method. The MATLAB 2013a provides the approximate solution.

Keywords: Nonlinear heat transfer; boundary intagral specification; implicit Euler method; higher order paralell splitting finite difference method; composite simpson quadrature; newton method.

AMS Subject Classification: 34A05, 34A08, 42A10, 40A30, 65B10

1 Introduction

This paper considers the problems of obtaining numerical solution to the one-dimensional nonlinear unsteady heat conduction equation [1],[2] given by

$$\begin{cases} \rho c_p \dfrac{\partial u}{\partial t} & = & \dfrac{\partial}{\partial x}\left(\kappa\left(u\right)\dfrac{\partial u}{\partial x}\right)\ a < x < b,\ \ 0 < t \leq T \\ \int_a^b u\left(x,t\right)dt & = & \alpha\left(t\right), 0 < t \leq T \\ \frac{\partial u}{\partial x}(b,t) & = & \beta\left(t\right), 0 < t \leq T \end{cases} \tag{1.1}$$

where the unknown function $u(x,t)$ is the temperature at position x and time t, ρ is the density, c_p is the specific heat capacity at constant pressure et κ is the thermal conductivity of the media. We assume that c_p and ρ have constant values, but κ depends on the temperature u. By Differentiating the first equation in right hand side of (1.1), we get

$$\rho c_p \frac{\partial u}{\partial t} = \kappa\left(u\right)\frac{\partial^2 u}{\partial x^2} + \partial_u \kappa\left(u\right)\left[\left(\frac{\partial u}{\partial x}\right)^2\right] \tag{1.2}$$

When κ does not depend on u i.e $\partial_u \kappa\left(u\right) = 0$, then (1.2) is a linear (parabolic) partial differential equation. When $\partial_u \kappa\left(u\right) \neq 0$, then (1.2) is nonlinear. The boundary conditions for equation (1.2) are the second and third equation in (1.1) where $\alpha\left(t\right)$ and $\beta\left(t\right)$ are known. The initial condition is assumed to be of the form

$$u(x,0) = g\left(x\right),\ x \in [a,b] \tag{1.3}$$

The first boundary condtion in (1.1) is the non-local condition and the second one is the Neumann condition for $x = b$.The nonlinear heat conduction equation with integral condition [3, 4] can model various phenomena in chemical, thermoelasticity, population dynamics, medical science, and so forth. These problems are studied by several authors [5, 6, 7]. Recently, there has been growing interest in developping computational techniques for their numerical solution [8, 9]. The recent paper, in [10], studied the equation (1.1)with Dirichlet boundary conditions. The authors, used the implicit Euler method for discretization in time. The result obtained is a sequence of nonlinear differential equations of order two in space, discretized by the second order centred finite difference method. Our work is an extend of the previous paper, using Neumann and non-local boundary conditions [11, 12, 13]. The Simpson's composite quadrature method is used to compute the integral in the boundary and a higher order parallel splitting finite difference method is applied to discretize the second order spatial derivative.

2 Method of Resolution

This section presents the Euler implicite discretization, the third-order accuracy finite difference scheme and the newton iteration method.

2.1 Implicite euler discretisation

We first discretise the equation (1.2) in time, using a time step $\tau >$. The time line $t \geq 0$ is partition by equally mesh-points as:

$$\tau_n = n\tau,\ n = 0, 1, 2, ... \tag{2.1}$$

Using the implicite Euler scheme [14], [10], equation (1.2) is dicretised on the mesh (2.1) as

$$\rho c_p \frac{u_n - u_{n-1}}{\tau} = \kappa\left(u_n\right) \frac{d^2 u_n}{dx^2} + \partial_u \kappa\left(u_n\right) \left(\frac{du_n}{dx}\right)^2 \tag{2.2}$$

where $u_n = u_n(x)$ and $u_{n-1} = u_{n-1}(x)$ approximate the values of $u(x, t_n)$ and $u(x\ t_{n-1})$ respectively. The equation (2.2) is the approximate of the partial differential equation (1.2). The error is $O(\tau)$, hence the discretisation scheme is first-order accurate in time. This implicite method is stable. Solving the equation (2.2), for the second spatial derivative of the temperature u_n, we get as in [10]:

$$\frac{d^2 u_n}{dx^2} = \frac{\rho c_p \left(u_n - u_{n-1}\right)}{\tau \kappa\left(u_n\right)} - \frac{\partial_u \kappa\left(u_n\right)}{\kappa\left(u_n\right)} \left(\frac{du_n}{dx}\right)^2$$

or

$$\frac{d^2 u_n}{dx^2} = \phi\left(u_n, v_n, u_{n-1}\right) \tag{2.3a}$$

where $v_n = \dfrac{du_n}{dx}$ is the temperature gradient and ϕ is the following nonlinear fonction

$$\phi\left(u_n, v_n, u_{n-1}\right) = \frac{\psi\left(u_n, v_n, u_{n-1}\right)}{\kappa\left(u_n\right)} \tag{2.4}$$

$$\psi\left(u_n, v_n, u_{n-1}\right) = \rho c_p \frac{u_n - u_{n-1}}{\tau} - \partial_u \kappa\left(u_n\right) v_n^2 \tag{2.5}$$

Equation $(2.3a)$, together with the boundary conditions

$$\int_a^b u_n(x)\, dx = \alpha\left(t_n\right) \tag{2.6}$$

$$\frac{du_n(b)}{dx} = \beta\left(t_n\right) \tag{2.7}$$

constitutes a nonlinear non-local problem for the unknown function u_n. Given the known function u_{n-1} , the problem can be solved, by some numerical technique for nonlinear problems. Starting from the initial condition u_0, we can solve successively $(2.3a)$ for $n = 1, 2, ...$The presence of an integral term in a boundary condition can be computed using the 1/3 Simpson quadrature approximation to convert non-local boundary value problem to a more desirable form. The accuracy of the quadrature must be compatible with that of the discretization of the differential equation.

2.2 Finite difference discretization method

We use the higher order parallel splitting finite difference method [15] ,[4], [16] for the solution of the problem $(2.3a)$.The intervals $[a, b]$ is partitionned by N equally separated mesh-points:

$$x_i = a + ih,\ i = 0, 1, 2, ..., N \tag{2.8}$$

where $h = \dfrac{b-a}{N}$. To approximate the space derivative in the equation $(2.3a)$ to third-order accuracy at some general point x on the uniform mesh (2.8) , assume that it may be replaced by the five point formula [17] :

$$\frac{d^2 u_n(x_i)}{dx^2} \simeq \frac{1}{12h^2}\left(11u_{n,i-1} - 20u_{n,i} + 6u_{n,i+1} + 4u_{n,i+2} - u_{n,i+3}\right), i = 1, ..., N-2 \tag{2.9}$$

$$\frac{d^2 u_n(x_{N-1})}{dx^2} \simeq \frac{1}{12h^2}\left(u_{n,i-3} - 6u_{n,i-2} + 26u_{n,i-1} - 4u_{n,i} + 21u_{n,i+1} - u_{n,i+2}\right) \tag{2.10}$$

$$\frac{d^2 u_n(x_N)}{dx^2} \simeq \frac{1}{12h^2}\left(2u_{n,i-4} - 11u_{n,i-3} + 24u_{n,i-2} - 14u_{n,i-1} + 10u_{n,i} - 9u_{n,i+1}\right) \tag{2.11}$$

We set $x = x_i$ in (2.3a), and then replace the values $u_n(x_i)$, $v_n(x_i)$, $u_{n-1}(x_i)$ with their approximations $u_{n,i}$, $v_{n,i}$, $u_{n-1,i}$ where

$$v_{n,i} = \frac{u_{n,i+1} - u_{n,i-1}}{2h} \tag{2.12}$$

Then, the equation (2.3a) can be written as

$$\frac{1}{12h^2}\left(11u_{n,i-1} - 20u_{n,i} + 6u_{n,i+1} + 4u_{n,i+2} - u_{n,i+3}\right) = \phi\left(u_{n,i}, v_{n,i}, u_{n-1,i}\right) \tag{2.13}$$

for $i = 1, ..., N-2$

2.3 Derivatives for the newton method

Let us define the function $\phi(u, v; w) = \frac{\psi(u,v;w)}{\kappa(u)}$and introduce the notation $\phi = \phi(u, v; w)$, $\psi = \psi(u, v; w)$. Denoting also the derivatives by $q = q(u, v; w)$, $p = p(u, w)$,we get

$$q = \frac{\partial \phi}{\partial u} = \frac{1}{\kappa(u)}\left[\frac{\partial \psi}{\partial u} - \phi \partial_u \kappa(u)\right]$$

$$p = \frac{\partial \phi}{\partial v} = \frac{1}{\kappa(u)}\frac{\partial \psi}{\partial v} \tag{2.14}$$

where

$$\frac{\partial \psi}{\partial u} = \frac{\rho c_p}{\tau} - \partial^2_{uu}\kappa(u)\, v^2 \tag{2.15}$$

$$\frac{\partial \psi}{\partial v} = -2\partial_u \kappa(u)\, v \tag{2.16}$$

2.4 Solving the nonlinear systems by newton method

The system of nonlinear algebric equations corresponding to (2.13),(2.6) ,(2.7) can be written as follow:

$$\begin{cases} -\alpha_1 u_{n,1} + \alpha_2 u_{n,2} + \alpha_3 u_{n,3} + \alpha_4 u_{n,4} + \sum_{j=5}^{N} \alpha_j u_{n,j} + \frac{11\alpha(t_n)}{hc_0} - 12h^2\phi_{n,1} & = 0 \\ 11u_{n,i-1} - 20u_{n,i} + 6u_{n,i+1} + 4u_{n,i+2} - u_{n,i+3} - 12h^2\phi_{n,i} & = 0 \\ u_{n,N-4} - 6u_{n,N-3} + 26u_{n,N-2} - 5u_{n,N-1} + 21u_{n,N} - 2h\beta(t_n) - 12h^2\phi_{n,N-1} & = 0 \\ 2u_{n,N-4} - 11u_{n,N-3} + 24u_{n,N-2} - 23u_{n,N-1} + 10u_{n,N} - 18h\beta(t_n) - 12h^2\phi_{n,N} & = 0 \end{cases}$$

where

$$\phi_{n,i} = \phi\left(u_{n,i}, v_{n,i}, u_{n-1,i}\right), \ 1 \le i \le N \tag{2.17}$$

$$c_0 = c_N = \frac{1}{3},\ c_{2j} = \frac{2}{3}, j = 1, 2, ..., \frac{N}{2} - 1,\ \ c_{2j-1} = \frac{4}{3},\ \ j = 1, ..., \frac{N}{2} \tag{2.18}$$

$$\alpha_1 = -20 - \frac{11c_1}{c_0}, \alpha_2 = 6 - \frac{11c_2}{c_0},$$

$$\alpha_3 = 4 - \frac{11}{c_0}c_3, \alpha_4 = -1 - \frac{11}{c_0}c_4, \alpha_N = -11,$$

$$\alpha_j = -\begin{cases} -44 & \text{when } j \text{ odd}, j = 5 \le j < N \\ -22 & \text{when } j \text{ even}, j = 6 \le j < N \end{cases}$$

This system of nonlinear equation can be written as $\mathrm{G}(\mathrm{u}_n) = 0$

$$\mathbf{G}_n(\mathbf{u}_n) = \begin{bmatrix} G_{n;1}(\mathbf{u}_n) \\ G_{n;2}(\mathbf{u}_n) \\ ... \\ G_{n;N}(\mathbf{u}_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ ... \\ 0 \end{bmatrix}, \qquad \mathbf{u}_n = \begin{bmatrix} u_{n,1} \\ u_{n,2} \\ ... \\ u_{n,N} \end{bmatrix} \tag{2.19}$$

with components

$$G_{n,1}\left(\mathbf{u}_n\right)=\alpha_1 u_{n,1}+\alpha_2 u_{n,2}+\alpha_3 u_{n,3}+\alpha_4 u_{n,4}+\sum_{j=5}^{N}\alpha_j u_{n,j}+\frac{11\alpha\left(t_n\right)}{hc_0}-12h^2\phi_{n,1}$$

$$G_{n;i}\left(\mathbf{u}_n\right)=11u_{n,i-1}-20u_{n,i}+6u_{n,i+1}+4u_{n,i+2}-u_{n,i+3}-12h^2\phi_{n,i},i=\ 2,...,N-2 \quad (2.20)$$

$$G_{n,N-1}\left(\mathbf{u}_n\right)=u_{n,N-4}-6u_{n,N-3}+26u_{n,N-2}-5u_{n,N-1}+21u_{n,N}-2h\beta\left(t_n\right)-12h^2\phi_{n,N-1}$$

$$G_{n,N}\left(\mathbf{u}_n\right)=2u_{n,N-4}-11u_{n,N-3}+24u_{n,N-2}-23u_{n,N-1}+10u_{n,N}-18h\beta\left(t_n\right)-12h^2\phi_{n,N}$$

Assume that the exact solution is $\mathbf{u}_n^{(e)}$. Suppose that the initial estimate of the solution is $\mathbf{u}_n^{(0)}$. A Talor developpement of first order is

$$\mathbf{G}_n\left(\mathbf{u}_n^{(e)}\right)=\mathbf{G}_n\left(\mathbf{u}_n^{(0)}\right)+\frac{\partial\mathbf{G}_n}{\partial\mathbf{u}_n}\left(\mathbf{u}_n^{(0)}\right)\Delta\mathbf{u}_n$$

with $\Delta\mathbf{u}_n=\mathbf{u}_n^{(e)}-\mathbf{u}_n^{(0)}$ By using

$$\mathbf{G}_n\left(\mathbf{u}_n^{(e)}\right)=0 \quad (2.21)$$

we obtain

$$\frac{\partial\mathbf{G}_n}{\partial\mathbf{u}_n}\left(\mathbf{u}_n^{(0)}\right)\Delta\mathbf{u}_n=-\mathbf{G}_n\left(\mathbf{u}_n^{(0)}\right)$$

This system of linear equations leads to new approximation $\mathbf{u}_n^{(1)}=\mathbf{u}_n^{(0)}+\Delta\mathbf{u}_n$. The Newton-Raphson Algorithm with the jacobian matrix $L_n^{(k)}$ is then

$$L_n^{(k)}\left(\mathbf{u}_n^{(k-1)}\right)\Delta\mathbf{u}_n^{(k)}=-\mathbf{G}_n\left(\mathbf{u}_n^{(k-1)}\right)$$

$$L_n^{(k)}\left(\mathbf{u}_n^{(k-1)}\right)=\frac{\partial\mathbf{G}_n}{\partial\mathbf{u}_n}\left(\mathbf{u}_n^{(k-1)}\right)$$

$$\mathbf{u}_n^{(k)}=\mathbf{u}_n^{(k-1)}+\Delta\mathbf{u}_n^{(k)}$$

Starting by some initial guess $\mathbf{u}_n^{(0)}$, the equation (2.21) can be solved by the Newton iterative method:

$$\mathbf{u}_n^{(k+1)}=\mathbf{u}_n^{(k)}-\ L_n^k\left(\mathbf{u}_n^{(k)}\right)^{-1}\mathbf{G}_n\left(\mathbf{u}_n^{(k)}\right),\ k=0,1,2,... \quad (2.22)$$

with the matrix L_n^k in the form

$$L_n^{(k)}=\frac{\partial\mathbf{G}_n}{\partial\mathbf{u}_n}=\begin{bmatrix}\frac{\partial\mathbf{G}_{n,1}}{\partial\mathbf{u}_{n,1}} & \frac{\partial\mathbf{G}_{n,1}}{\partial\mathbf{u}_{n,2}} & ... & \frac{\partial\mathbf{G}_{n,1}}{\partial\mathbf{u}_{n,N}}\\ \frac{\partial\mathbf{G}_{n,2}}{\partial\mathbf{u}_{n,1}} & \frac{\partial\mathbf{G}_{n,2}}{\partial\mathbf{u}_{n,2}} & ... & \frac{\partial\mathbf{G}_{n,2}}{\partial\mathbf{u}_{n,N,}}\\ ... & ... & ... & \frac{\partial\mathbf{G}_{n}}{\partial\mathbf{u}_{n}}\\ \frac{\partial\mathbf{G}_{n},N}{\partial\mathbf{u}_{n,1}} & \frac{\partial\mathbf{G}_{n,N}}{\partial\mathbf{u}_{n,2}} & ... & \frac{\partial\mathbf{G}_{n,N}}{\partial\mathbf{u}_{n,N}}\end{bmatrix} \quad (2.23)$$

The elements of jacobian, are

$$L_n^k\left(1,1\right)=\alpha_1-12h^2q_{n,1}^{(k)},L_n^k\left(1,2\right)=\alpha_2-6hp_{n,2}^{(k)},L_n^k\left(1,3\right)=\alpha_3,L_n^k\left(1,4\right)=\alpha_4$$

$$L_n^k\left(1,i\right)=\begin{cases}-44 & \text{when } i \text{ odd}\\ -22 & \text{when } i \text{ even}\end{cases}$$

$$L_n^{(k)}\left(i,i\right)=-20-12h^2q_{n,i}^{(k)},i=2,3,...,N-1$$

$$L_n^{(k)}\left(i,i-1\right)\ =\ 11+-6hp_{n,i}^{(k)},$$

$$\begin{aligned} L_n^{(k)}(i,i+1) &= -6--6hp_{n,i}^{(k)},\ L_n^{(k)}(i,i+2)=4, \\ L_n^{(k)}(i,i+3) &= -1; \end{aligned} \tag{2.24}$$

$$\begin{aligned} L_n^{(k)}(N-1,N-1) &= 5-12h^2q_{n,N-1}^{(k)} \\ L_n^{(k)}(N-1,N-2) &= 26+6hp_{n,N-1}^{(k)} \\ L_n^{(k)}(N-1,N) &= 21-6hp_{n,N-1}^{(k)} \end{aligned}$$

$$L_n^{(k)}(N-1,N-4)=1;\quad L_n^{(k)}(N-1,N-3)=-6$$

$$L_n^{(k)}(N,N)=10-12h^2q_{n,N}^{(k)};\ L_n^{(k)}(N,N-1)=-23+6hp_{n,N}^{(k)};$$

$$L_n^{(k)}(N,N-2)=24;\ L_n^{(k)}(N,N-3)=-11;\ L_n^{(k)}(N,N-4)=2$$

where

$$q_{n,i}^{(k)}=q\left(u_{n,i}^{(k)},v_{n,i}^{(k)};u_{n-1,i}\right)\ p_{n,i}^{(k)}=p\left(u_{n,i}^{(k)},v_{n,i}^{(k)}\right). \tag{2.25}$$

The iteration (2.22) is one step(two-level) iteration. Given an initial guess $\mathbf{u}_n^{(0)}$, we can calculate next approximation $\mathbf{u}_n^{(k+1)}$, $k=0,1,2,...$, using (2.22). The limiting vector $\mathbf{u}_n=\lim_{k\to+\infty}\left(\mathbf{u}_n^{(k+1)}\right)$ is a solution to the nonlinear system (2.21), if the sequence is convergent. The iteration process is ended when

$$\left\|\mathbf{u}_n^{(k+1)}-\mathbf{u}_n^{(k)}\right\|<\varepsilon \tag{2.26}$$

This inequality is called a stopping criteria. Generally, we use as $\mathbf{u}_n^{(0)}$, the solution $\mathbf{u}_{n-1}$ found in previous step.

3 Computer Experiment

Consider a thin homogeneous rod, along the x-axis between the point x=1 and x=3, without heat source and without radiation. The density ρ and the heat capacity c_p are constant, but the thermal conductibility κ depends on the temperature [10] as

$$\kappa=\kappa_0\exp(\chi u) \tag{3.1}$$

Such a temperature dependence occurs in real physical systems, e.g. for silicon [1]. We choose $\rho=1$, $c_p=1$, $\kappa_0=.1$. The boundary condtions are

$$\int_1^3 u(x,t)\,dx=\frac{5}{3},\ t>0 \tag{3.2}$$

$$\frac{\partial u(3,t)}{\partial x}=\frac{3}{2},t>0 \tag{3.3}$$

The initial temperature profil is [10]

$$u(x,0)=2-\frac{x-1}{2}+(x-1)(x-3),\ x\in[1,\ 3] \tag{3.4}$$

We solve the partial differential equation (1.1)with boundary condition (3.2)and (3.3) and the initial condition (3.4) by the method described in this paper to determine the time evolution of (3.4). The step size is choosing to be $\tau=0.5$ in the integration range $0\le t\le 15$.We discretised the interval [1, 3] with $N=41$ mesh-points, i.e $h=0.05$. The equation is solved for the parameter $\chi=0.0,\ 0.05, 0.5,\ 1.0, 1.5, 2.0$.

The linear steady profil correspond to $\chi=0$. The final distribution temperature for this value is pratically 10. This solution is obtained by putting $\frac{\partial u}{\partial t}=0$.For $\chi=0.05$, the vfinal temperature in the experiment are between 4 and 6. For $\chi=0.5$, the final distribution is between 2 and 3.For $\chi=1$, the final distribution is betweel 1, 5 and 2. These values converge towards the steady distribution wich is 10. For $\chi=1.5$ the value is about 1, 45 between 1 and 1, 5. For $\chi=2$ the final distribution value is about 1, 25 between 1 and 1, 5. The convergence toward the steady solution is low.

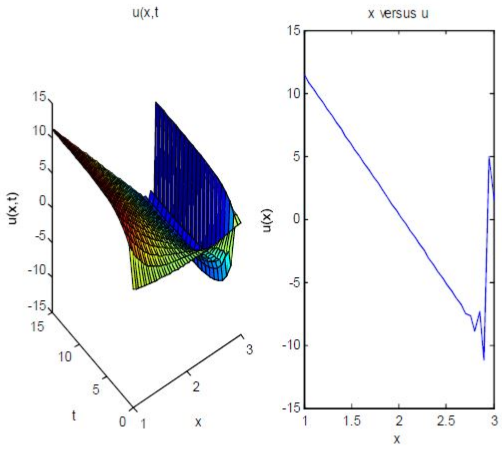

Fig. 1. $\chi = 0$

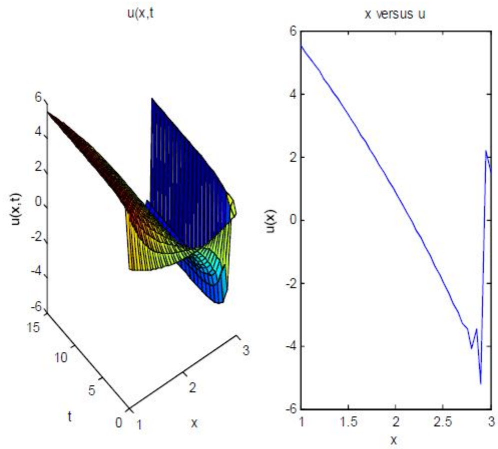

Fig. 2. $\chi = 0.05$

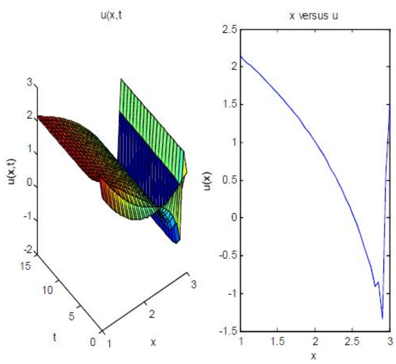

Fig. 3. $\chi = 0.5$

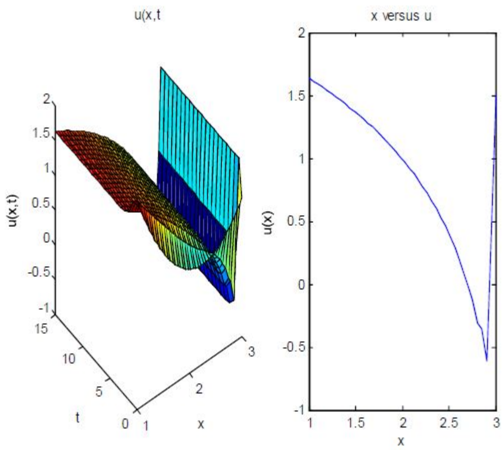

Fig. 4.$\chi = 1$

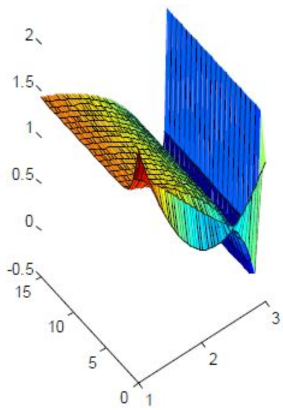

Fig. 5.$\chi = 1.5$

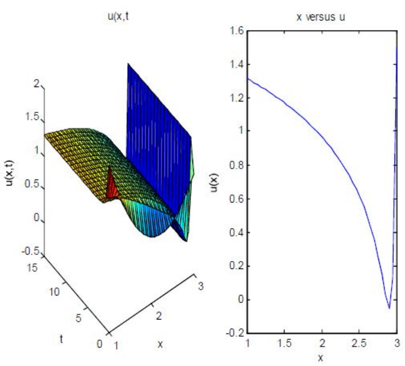

Fig. 6.$\chi = 2$

4 Conclusion

The paper consider nonlinear heat transfer problem with non local boundary condition and temperature dependent thermal conductivity. This one-dimensional unsteady heat conduction equation was solved numerically by using implicit time-discretization and third-order accuracy finite difference method. The boundary integral specification was computed using the Simpson's composite $\frac{1}{3}$ quadrature. Newton method and MATLAB program provided the solution of the arising nonlinear two-point boundary value problems. The results obtained by the numerical computer experiments are consistent with the expected experimental datta. The proposed method is stable.

Competing Interests

Authors have declared that no competing interests exist.

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