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V_4 Magic Labelings of Some Graphs

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Abstract

Let A be an abelian group with identity element 0. A graph $G = (V, E)$ is said to admit an a-sum A-magic labeling if there exists an edge labeling $\ell : E(G) \longrightarrow A \setminus \{0\}$ and $a \in A$ such that the induced vertex labeling $\ell^+ : V(G) \longrightarrow A$ defined by

$$\ell^+(u) = \sum\{\ell(uv) : uv \in E(G)\}$$

is the constant map, $\ell^+(u) = a$ for all $u \in V(G)$. If $a = 0$, the labeling ℓ is called a zero-sum A-magic labeling of G. A graph G is said to be a-sum (resp.zero-sum) A-magic if G admits an a-sum (resp.zero-sum) A-magic labeling. In this paper we will consider the Klein 4 group $V_4 = \{0, a, b, c\} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and investigate graphs that are a-sum A-magic, zero-sum A-magic and both a-sum and zero-sum A-magic.

Keywords: V_4 magic graph; a-sum V_4 magic graph; zero-sum V_4 magic graph.

2010 Mathematics Subject Classification: 05C78; 05C25.

1 Introduction

In this paper all graphs are connected,finite,simple and undirected. For graph theoretic notations and terminology not directly defined in this paper, we refer to readers [1].

For an abelian group A, written additively, any mapping $\ell : E(G) \longrightarrow A \setminus \{0\}$ is called a labeling,

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where 0 denote the identity element in A. For any abelian group A, a graph $G = (V, E)$ is said to be A-magic if there exists a labeling $\ell : E(G) \longrightarrow A \setminus \{0\}$ such that the induced vertex set labeling $\ell^+ : V(G) \longrightarrow A$ defined by

$$\ell^+(u) = \sum \{\ell(uv) : uv \in E(G)\}$$

is a constant map [2]. Observe that A-magic labeling of a graph need not be unique. The V_4 magic graphs was first introduced by S. M. Lee et al. in 2002 [2]. There has been an increasing interest in the study of V_4 magic graphs since the publication of [2].

We follow the following definitions and notations described in our earlier publications [3, 4]. A V_4 magic graph G is called a-sum V_4 magic labeling of G, if there exists a labeling $\ell : E \to V_4 \setminus \{0\}$ such that $\ell^+(v) = a$ for all $v \in V$. Any graph that admits an a-sum V_4 magic labeling is called an a-sum V_4 magic graph. When $a = 0$, we call G a zero-sum V_4 magic graph.

(i) $\mathscr{V}_a$, the class of a-sum V_4 magic graphs,

(ii) $\mathscr{V}_0$, the class of zero-sum V_4 magic graphs,and

(iii) $\mathscr{V}_{a,0}$, the class of graphs which are both a-sum and zero -sum V_4 magic.

In this paper, we investigate a class of graphs that belongs to the above categories.

2 Main Theorems

Definition 2.1. The Jahangir graph $J_{n,m}$ for $m \geq 3$ is a graph consisting of a cycle C_{nm} with one additional vertex called the central vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm}.

Observe that $J_{n,m}$ has $nm + 1$ vertices. The Jahangir graph $J(2, 8)$ is shown in Fig. 1.

Lemma 2.1. *If $\ell : E(J_{n,m}) \longrightarrow V_4 \setminus \{0\}$ is an a-sum V_4 magic labeling of $J_{n,m}$, then*

$$\sum_{i=1}^{m} \ell^+(u_i) + \sum_{i=1}^{m} \sum_{j=1}^{n-1} \ell^+(v_{ij}) + \ell^+(w) = 0$$

where $u_1, u_2, \cdots u_m$ are the m vertices of C_{nm} which is adjacent to the central vertex w and $v_{i1}, v_{i2}, \cdots v_{i(n-1)}$ are the $(n-1)$ vertices between u_i and $u_{i+1}, i = 1, 2, \cdots m$ where $u_{m+1} = u_1$.

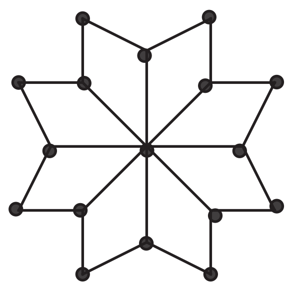

Fig. 1. The Jahangir graph $J_{2,8}$

Proof. We have

$$\ell^{+}(w) = \sum_{i=1}^{m} \ell(u_i w). \tag{1}$$

For $i = 1, 2, \cdots m,$ we have

$$\ell^{+}(u_i) = \ell(u_i w) + \ell(u_i v_{i1}) + \ell(u_i v_{(i-1)(n-1)}), \tag{2}$$

$$\ell^{+}(v_{ij}) = \ell(v_{ij} v_{i(j+1)}) + \ell(v_{i(j-1)} v_{ij}), j = 2, 3, \cdots n-2, \tag{3}$$

$$\ell^{+}(v_{i1}) = \ell(u_i v_{i1}) + \ell(v_{i1} v_{i2}), \tag{4}$$

$$\ell^{+}(v_{i(n-1)}) = \ell(v_{i(n-1)} u_{i+1}) + \ell(v_{i(n-1)} v_{i(n-2)}). \tag{5}$$

From equations (2),(4),(5) we get

$$\sum_{i=1}^{m} \ell^{+}(u_i) + \sum_{i=1}^{m} \ell^{+}(v_{i1}) + \sum_{i=1}^{m} \ell^{+}(v_{i(n-1)}) = \ell^{+}(w) + \sum_{i=1}^{m} \ell(v_{i1} v_{i2}) + \sum_{i=1}^{m} \ell(v_{i(n-1)} v_{i(n-2)}). \tag{6}$$

From equation (3) we get,

$$\sum_{i=1}^{m} \sum_{j=2}^{n-2} \ell^{+}(v_{ij}) = \sum_{i=1}^{m} \sum_{j=2}^{n-2} \ell(v_{ij} v_{i(j+1)}) + \sum_{i=1}^{m} \sum_{j=2}^{n-2} \ell(v_{i(j-1)} v_{ij}). \tag{7}$$

Adding the equations (6) and (7) we obtain,

$$\sum_{i=1}^{m} \ell^{+}(u_i) + \sum_{i=1}^{m} \sum_{j=1}^{n-1} \ell^{+}(v_{ij}) = \ell^{+}(w).$$

This implies that,

$$\sum_{i=1}^{m} \ell^{+}(u_i) + \sum_{i=1}^{m} \sum_{j=1}^{n-1} \ell^{+}(v_{ij}) + \ell^{+}(w) = 0.$$

This completes the proof of the lemma. □

Theorem 2.2. *$J_{n,m} \in \mathscr{V}_a$ if and only if both n and m are odd.*

Proof. Let the vertices of $J_{n,m}$ be as in the proof of Lemma 2.1. Also let $u_i = u_{i(mod m)}$ and $v_{ij} = v_{i(mod m) j(mod n)}$. First assume that $J_{n,m} \in \mathscr{V}_a$. Then we have $(nm + 1)a = 0$. This implies that $nm + 1$ is even which in turn implies that both n and m are odd. Conversely, assume that n and m are odd. Define a labeling $\ell : E(J_{n,m}) \longrightarrow V_4 \setminus \{0\}$ as follows.

$$\begin{aligned}
&\text{For } i = 1, 2, \cdots m \text{ do}: \\
&\qquad \ell(u_i w) = a, \\
&\qquad \ell(u_i v_{i1}) = b, \\
&\qquad \text{end for} \\
&\qquad \ell(u_1 v_{m(n-1)}) = b, \\
&\qquad \ell(u_i v_{(i-1)(n-1)}) = b, i = 2, 3, \cdots m \\
&\text{For } i = 1, 2, \cdots m \text{ do}: \\
&\qquad \ell(v_{ij} v_{i(j+1)}) = c, j = 1, 3, \cdots n-2, \\
&\qquad \ell(v_{ij} v_{i(j+1)}) = b, j = 2, 4, \cdots n-3. \\
&\qquad \text{end for}
\end{aligned}$$

With this labeling we get

$$\ell^{+}(w) = ma = a,$$
$$\ell^{+}(u_i) = b + b + a = a,$$
$$\ell^{+}(v_{ij}) = b + c = a.$$

Obviously, ℓ is an a-sum V_4 magic labeling of $J_{n,m}$. □

Theorem 2.3. *$J_{n,m} \in \mathscr{V}_0$ for all n and m.*

Proof. Let the vertices of $J_{n,m}$ be as in the proof of Theorem 2.2. We consider the following cases.
Case 1: m even

Define a labeling $\ell : E(J_{n,m}) \longrightarrow V_4 \setminus \{0\}$ as follows.

$$\ell(u_i w) = c, \text{for } i = 1, 2, \cdots m,$$
$$\text{For } j = 1, 2, \cdots n-2 \text{ do}:$$
$$\ell(v_{ij} v_{i(j+1)}) = a, i = 1, 3, \cdots m-1,$$
$$\ell(v_{ij} v_{i(j+1)}) = b, i = 2, 4, \cdots m.$$
$$\text{end for}$$
$$\ell(u_i v_{i1}) = a, \text{for } i = 1, 3, \cdots m-1,$$
$$\ell(u_i v_{i1}) = b, \text{for } i = 2, 4, \cdots m,$$
$$\ell(u_{i+1} v_{i(n-1)}) = \begin{cases} a, & i = 1, 3, \cdots m-1 \\ b, & i = 2, 4, \cdots m+1 \end{cases}$$

Obviously, ℓ is a 0-sum V_4 magic labeling of $J_{n,m}$.

Case 2: m odd

Define a labeling $\ell : E(J_{n,m}) \longrightarrow V_4 \setminus \{0\}$ as follows.

$$\ell(u_i w) = a, \text{for } i = 1, 4, 5, \cdots m,$$
$$\ell(u_2 w) = b,$$
$$\ell(u_3 w) = c,$$
$$\text{For } j = 1, 2, \cdots n-2 \text{ do}:$$
$$\ell(v_{2j} v_{2(j+1)}) = a,$$
$$\ell(v_{ij} v_{i(j+1)}) = b, \text{for } i = 3, 5, \cdots m,$$
$$\ell(v_{ij} v_{i(j+1)}) = c, \text{for } i = 4, 6, \cdots m-1, m+1.$$
$$\text{end for}$$
$$\ell(u_2 v_{21}) = \ell(u_3 v_{2(n-1)}) = a,$$
$$\ell(u_i v_{i1}) = \ell(u_{(i+1)} v_{i(n-1)}) = b, \text{for } i = 3, 5, \cdots m,$$
$$\ell(u_i v_{i1}) = \ell(u_{(i+1)} v_{i(n-1)}) = c, \text{for } i = 4, 6, \cdots m-1, m+1.$$

Obviously, ℓ is a 0-sum V_4 magic labeling of $J_{n,m}$. □

Theorem 2.4. *If both m and n are odd, $J_{n,m} \in \mathscr{V}_{a,0}$.*

Proof. The proof follows from Theorems 2.2 and 2.3. □

Definition 2.2. The windmill graph $D_n^{(m)}$ is the graph obtained by taking m copies of the complete graph K_n with a vertex in common.

The windmill graph $D_3^{(4)}$ is shown in Fig. 2. The graph $D_3^{(m)}$ is called the Dutch windmill graph or the friendship graph, F_m.

Lemma 2.5. *If $\ell : E(D_n^{(m)}) \longrightarrow V_4 \setminus \{0\}$ is an a-sum V_4 magic labeling of $D_n^{(m)}$, then*

$$\sum_{i=1}^{m}\sum_{j=1}^{n-1} \ell^+(u_j^i) = \ell^+(v)$$

where $u_1^i, u_2^i, \cdots u_{(n-1)}^i$ are the vertices of i^{th} copy of K_n in $D_n^{(m)}$ and v is the common vertex.

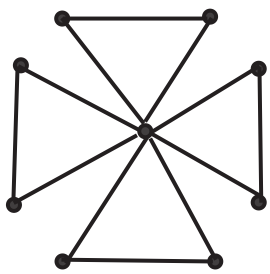

Fig. 2. Windmill graph $D_3^{(4)}$

Proof. The proof is similar to Lemma 2.1 . □

Theorem 2.6. *The windmill graph $D_n^{(m)} \in \mathscr{V}_a$ if and only if m is odd and n is even.*

Proof. Suppose $D_n^{(m)} \in \mathscr{V}_a$. Then by Lemma 2.5, $[m(n-1)+1]a = 0$. This implies that $m(n-1)$ is odd. This holds only when m is odd and n is even. Conversely suppose that m is odd and n is even. Let the vertices of $D_n^{(m)}$ be as in the Lemma 2.5. Define a labeling $\ell : E(D_n^{(m)}) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{gathered}\text{For } i = 1, 2, \cdots m \text{ do :}\\ \ell(u_j^i v) = a, j = 1, 2, \cdots n,\\ \ell(u_j^i u_{j+1}^i) = a, j = 1, 2, \cdots n.\\ \text{end for}\end{gathered}$$

Obviously ℓ is an a-sum V_4 magic labeling of $D_n^{(m)}$. □

Theorem 2.7. $D_n^{(m)} \in \mathscr{V}_0$ *for all n and m.*

Proof. We consider the following cases.

Case 1: n is odd.

Label all the edges by a. Then we have $\ell^+(v) = 0$ for all $v \in V(D_n^{(m)})$.

Case 2: n is even.

Define a labeling $\ell : E(D_n^{(m)}) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} &\text{For } i = 1, 2, \cdots m \text{ do :} \\ \ell(u_j^i u_{j+1}^i) &= a, j = 1, 3, \cdots n-1, \\ \ell(u_j^i u_{j+1}^i) &= b, j = 2, 4, \cdots n, \\ \ell(u_j^i u_k^i) &= c, j, k = 1, 2, \cdots n,\ k \neq j+1. \\ &\text{end for} \end{aligned}$$

Thus we get $\ell^+(v) = 0$ for all $v \in V(D_n^{(m)})$. Obviously ℓ is a zero-sum V_4 magic labeling of $D_n^{(m)}$. This completes the proof of the theorem. □

Theorem 2.8. *$D_n^{(m)} \in \mathscr{V}_{a,0}$ if and only if m is odd and n is even.*

Proof. The proof follows from theorems 2.6 and 2.7. □

Theorem 2.9. *$F_m \notin \mathscr{V}_a$ for any m.*

Proof. Observe that F_m is the one-point union of m copies of a rooted triangle. Let the vertices of the i^{th} copy be $0, u_i$ and v_i. Assume that 0 is the root of the triangles. If F_m admits an a-sum V_4 magic labeling, then

$$\ell^+(u_i) = \ell^+(v_i) = a.$$

This implies that for all i,

$$\ell(u_i v_i) = b,\ \ell(0u_i) = \ell(0v_i) = c$$
$$\text{or}\ \ \ell(u_i v_i) = c,\ \ell(0u_i) = \ell(0v_i) = b.$$

In both the cases, $\ell^+(0) = 2ma = 0$. This is a contradiction. □

Theorem 2.10. *$F_m \in \mathscr{V}_0$ for all m.*

Proof. Label all the edges by a. Obviously this is a zero-sum V_4 magic labeling of F_m. □

Theorem 2.11. *$F_m \notin \mathscr{V}_{a,0}$ for any m.*

Proof. The proof follows from theorems 2.9 and 2.10. □

Definition 2.3. (see [5]) The graph $P_2 \Box P_n$ is called a ladder. It is denoted by L_n.

Theorem 2.12. *Ladders L_n are a-sum V_4 magic for all n.*

Proof. Let $u_1, u_2, \cdots u_n$ and $v_1, v_2, \cdots v_n$ be the vertices of a ladder L_n such that $E(G) = \{u_i u_{(i+1)}/i = 1, 2, \cdots n-1\} \cup \{v_j v_{(j+1)}/j = 1, 2, \cdots n-1\} \cup \{u_i v_i/i = 1, 2, \cdots n\}$. Define a labeling $\ell : E(L_n) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} \ell(u_1 v_1) = \ell(u_n v_n) &= b, \\ \ell(u_i v_i) &= a, \text{for } i = 2, 3, \cdots n-1, \\ \ell(u_i u_{(i+1)}) = \ell(v_i v_{(i+1)}) &= c, \text{for } i = 1, 2, \cdots n-1. \end{aligned}$$

Then clearly ℓ is an a-sum V_4 magic labeling of L_n. □

Theorem 2.13. *(see [5]) $L_n \in \mathscr{V}_0$ for all n.*

Theorem 2.14. *$L_n \in \mathscr{V}_{a,0}$ for all n.*

Proof. The proof follows from theorems 2.12 and 2.13. □

Definition 2.4. (see [5]) The graph G with the vertex set $\{u_0, u_1, \cdots u_{n+1}, v_0, v_1, \cdots v_{n+1}\}$ and the edge set $\{u_i u_{i+1}, v_i v_{i+1} : 0 \leq i \leq n\} \bigcup \{u_i v_i/i = 1, 2, \cdots n\}$ is called ladder L_{n+2}.

Theorem 2.15. *(see [5]) $L_{n+2} \in \mathscr{V}_a$ for all n.*

Theorem 2.16. *$L_{n+2} \notin \mathscr{V}_0$ for any n.*

Proof. Since the graph has pendant edges it is not zero-sum V_4 magic for any n. □

Definition 2.5. (see [5]) The graph G with the vertex set $\{u_1, u_2, \cdots u_n, v_1, v_2, \cdots v_n\}$ and edge set $\{u_i u_{(i+1)}, v_i v_{(i+1)}, v_i u_{(i+1)} : 1 \leq i \leq n-1\} \cup \{u_i v_i : 1 \leq i \leq n\}$ is called a semiladder of length n.

Theorem 2.17. *Semiladders are a-sum V_4 magic for all n.*

Proof. Let G be a semiladder of length n. We consider two cases.
Case 1: n odd

Define a labeling $\ell : E(G) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} \ell(u_1 v_1) &= b, \\ \ell(u_i v_i) &= a, \text{for } i = 2, 3, \cdots n-1, \\ \ell(u_n v_n) &= b, \\ \ell(v_i u_{(i+1)}) &= a, \text{for } i = 1, 2, \cdots n-1. \\ &\text{For } i = 1, 3, \cdots n-2 \text{ do :} \\ \ell(u_i u_{(i+1)}) &= c, \\ \ell(v_i v_{(i+1)}) &= b. \\ &\text{end for} \\ &\text{For } i = 2, 4, \cdots n-1 \text{ do :} \\ \ell(u_i u_{(i+1)}) &= b, \\ \ell(v_i v_{(i+1)}) &= c. \\ &\text{end for} \end{aligned}$$

Thus ℓ is an a-sum V_4 magic labeling of G.

Case 2: n even

Define a labeling $\ell : E(G) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned}
\ell(u_1v_1) &= b,\\
\ell(u_iv_i) &= a, \text{for } i = 2, 3, \cdots n-1,\\
\ell(u_nv_n) &= c,\\
\ell(v_iu_{(i+1)}) &= a, \text{for } i = 1, 2, \cdots n-1,\\
&\text{For } i = 1, 3, \cdots n-1 \text{ do :}\\
\ell(u_iu_{(i+1)}) &= c,\\
\ell(v_iv_{(i+1)}) &= b.\\
&\text{end for}\\
&\text{For } i = 2, 4, \cdots n-2 \text{ do :}\\
\ell(u_iu_{(i+1)}) &= b,\\
\ell(v_iv_{(i+1)}) &= c.\\
&\text{end for}
\end{aligned}$$

Thus ℓ is an a-sum V_4 magic labeling of G. □

Theorem 2.18. *(see [5]) Semiladders are zero-sum V_4 magic for all n.*

Theorem 2.19. *If G is a semiladder, then $G \in \mathscr{V}_{a,0}$.*

Proof. The proof follows from theorems 2.17 and 2.18. □

Definition 2.6. (see [5]) Composition of two graphs $G[H]$ has $V(G) \times V(H)$ as vertex set in which (g_1, h_1) is adjacent to (g_2, h_2) whenever $g_1g_2 \in E(G)$ or $g_1 = g_2$ and $h_1h_2 \in E(H)$.

The graph $P_4[K_2^c]$ is shown in Fig. 3.

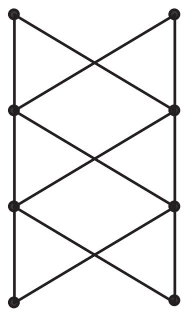

Fig. 3. $P_4[K_2^c]$

Theorem 2.20. *The composition $P_n[K_2^c]$ is a-sum V_4 magic for all n.*

Proof. Let $v_1, v_2, \cdots v_n$ be the vertices of P_n and x, y be that of K_2^c. Let u_i denote the vertices (v_i, x) and w_i denote (v_i, y) of $P_n[K_2^c], 1 \leq i \leq n$. Define a labeling $\ell : E(P_n[K_2^c]) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned}
\ell(u_i u_{(i+1)}) &= b, \text{for } i = 1, 2, \cdots n-1,\\
\ell(w_1 w_2) &= b,\\
\ell(w_i w_{(i+1)}) &= c, \text{for } i = 2, 3, \cdots n-1,\\
\ell(u_1 w_2) &= c,\\
\ell(u_i w_{(i+1)}) &= b, \text{for } i = 2, 3, \cdots n-1,\\
\ell(u_{(i+1)} w_i) &= c, \text{for } i = 1, 2, \cdots n-1.
\end{aligned}$$

Thus ℓ is an a-sum V_4 magic labeling of $P_n[K_2^c]$. □

Theorem 2.21. *(see [5]) $P_n[K_2^c] \in \mathscr{V}_0$ for all n.*

Theorem 2.22. *$P_n[K_2^c] \in \mathscr{V}_{a,0}$ for all n.*

Proof. The proof follows from theorems 2.20 and 2.21. □

Definition 2.7. (see [5]) One point union of any number of connected graphs is obtained by identifying one vertex from each graph. One point union of t cycles each of length n is denoted by $C_n^{(t)}$.

Lemma 2.23. *If $\ell : C_n^{(t)} \longrightarrow V_4 \setminus \{0\}$ is an a-sum magic labeling of $C_n^{(t)}$, then*

$$\sum_{i=1}^{n-1} \sum_{j=1}^{t} \ell^+(u_{ij}) = \ell^+(v)$$

where $u_{1j}, u_{2j}, \cdots u_{(n-1)j}$ are the vertices of j-th copy of C_n and v is the common vertex.

Proof. The proof is similar to Lemma 2.1. □

Theorem 2.24. *$C_n^{(t)} \in \mathscr{V}_a$ if and only if n is even and t is odd.*

Proof. First assume that $C_n^{(t)} \in V_a$. Then by Lemma 2.23, $[(n-1)t + 1]a = 0$. This equation holds if and only if n is even and t is odd. Conversely suppose that n is even and t is odd. Define a labeling $\ell : C_n^{(t)} \longrightarrow V_4 \setminus \{0\}$ as follows

$$\begin{aligned}
&\text{For } j = 1, 2, \cdots t \text{ do :}\\
\ell(u_{ij} u_{(i+1)j}) &= b, \text{for } i = 1, 3, \cdots n-1,\\
\ell(u_{ij} u_{(i+1)j}) &= c, \text{for } i = 2, 4, \cdots n.\\
&\text{end for}
\end{aligned}$$

Obviously $\ell^+(u_{ij}) = a,\ \ell^+(v) = a$. □

Theorem 2.25. *(see [5]) $C_n^{(t)} \in \mathscr{V}_0$ for all n and t.*

Theorem 2.26. *$C_n^{(t)} \in \mathscr{V}_{a,0}$ if and only if n is even and t is odd.*

Proof. The proof follows from theorems 2.24 and 2.25. □

Definition 2.8. (see [5]) A snake graph is formed by taking n-copies of a cycle C_m and identifying exactly one edge of each copy to a distinct edge of the path $P_{(n+1)}$, which is called as the backbone of the snake. It is denoted by $T_n^{(m)}$.

The graph $T_n^{(m)}$ is shown in Fig. 4.

Theorem 2.27. *$T_n^{(m)} \in \mathscr{V}_a$ if and only if m is even and n is odd.*

Proof. Suppose that $T_n^{(m)} \in V_a$. Let $u_{ij}, i = 1, 2, \cdots n, j = 1, 2, \cdots m$ be the vertices in the graph. Without loss of generality assume that $u_{(n+1)j} = u_{1j}$. Then we have

$$\sum_{i=2}^{n}\sum_{j=2}^{m} \ell^+(u_{ij}) + \sum_{j=1}^{m} \ell^+(u_{1j}) = 0$$

This implies that $[(m-1)n+1]$ is even which again implies that m is even and n is odd. Conversely assume that m is even and n is odd. Define a labeling $\ell : E(T_n^{(m)}) \longrightarrow V_4 \setminus \{0\}$ by

$$\text{For } i = 1, 3, \cdots n \text{ do :}$$

$$\ell(u_{ij}u_{(j+1)}) = \begin{cases} b, & j = 1, 3, \cdots m-1 \\ c, & j = 2, 4, \cdots n \end{cases}$$

$$\text{end for}$$

$$\text{For } i = 2, 4, \cdots n-1 \text{ do :}$$

$$\ell(u_{ij}u_{(j+1)}) = \begin{cases} c, & j = 2, 3, \cdots m-2 \\ b, & j = 1, m-1, m \end{cases}$$

$$\text{end for}$$

Clearly $\ell^+(v) = a$ for all $v \in V(T_n^{(m)})$. □

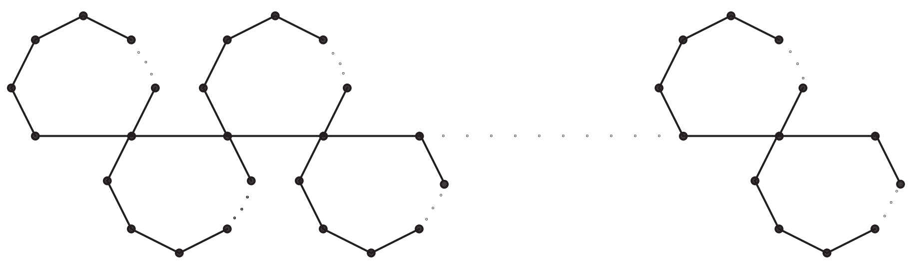

Fig. 4. Snake Graph $T_n^{(m)}$

Theorem 2.28. *(see [5]) $T_n^{(m)} \in \mathscr{V}_0$ for all n and m.*

Theorem 2.29. *$T_n^{(m)} \in \mathscr{V}_{a,0}$ if and only if m is even and n is odd.*

Proof. The proof follows from theorems 2.27 and 2.28. □

Definition 2.9. (see [5]) The cartesian product of graphs P_m and P_n denoted, $P_m \square P_n$ is called a planar grid.

The planar grid $P_6 \square P_4$ is shown in Figure 2.

Theorem 2.30. *The planar grid $P_m \square P_n$ is a-sum V_4 magic if and only if mn is even.*

Proof. Suppose that $P_m \square P_n \in \mathscr{V}_a$. Let $(i,j), i = 0,1,\cdots m-1, j = 0,1,\cdots n-1$ denote the vertices of $P_m \square P_n$. We have $\sum_{i=1}^{m-1}\sum_{j=1}^{n-1}\ell^+((i,j)) = 0$. Thus we have mn is even. For the converse consider the following cases.
Case 1: Both m and n are even.

Define a labeling $\ell : E(P_m \square P_n) \longrightarrow V_4 \setminus \{0\}$ as follows.

$$
\begin{aligned}
&\text{For } i = 0,1,\cdots m-2 \text{ do :}\\
\ell((i,j)(i+1,j)) &= b, j = 0, n-1\\
&\text{end for}\\
&\text{For } j = 0,1,\cdots n-2 \text{ do :}\\
\ell((i,j)(i,j+1)) &= c, i = 0, m-1\\
&\text{end for}\\
&\text{For } i = 1,2,\cdots m-2 \text{ do :}\\
\ell((i,j)(i,j+1)) &= \begin{cases} a, & j = 0,2,\cdots n-2\\ c, & j = 1,3,\cdots n-3 \end{cases}\\
&\text{end for}\\
&\text{For } j = 1,2,\cdots n-2 \text{ do :}\\
\ell((i,j)(i+1,j)) &= \begin{cases} a, & i = 0,2,\cdots m-2\\ b, & j = 1,3,\cdots m-3 \end{cases}\\
&\text{end for}
\end{aligned}
$$

With this labeling $P_m \square P_n$ is a-sum V_4 magic.

Case 2: m is even and n is odd.

Define a labeling $\ell : E(P_m \square P_n) \longrightarrow V_4 \setminus \{0\}$ as follows.

$$
\begin{aligned}
\ell((i,0)(i+1,0)) &= b,\ i = 0,1,\cdots m-2\\
\ell((i,n-1)(i+1,n-1)) &= \begin{cases} b, & i = 0,2,\cdots m-2\\ a, & i = 1,3,\cdots m-3 \end{cases}\\
&\text{For } j = 0,1,\cdots n-2 \text{ do :}\\
\ell((i,j)(i,j+1)) &= c,\ i = 0, m-1\\
&\text{end for}\\
&\text{For } j = 1,2,\cdots n-2 \text{ do :}\\
\ell((i,j)(i+1,j)) &= \begin{cases} a, & i = 0,2,\cdots m-2\\ c, & i = 1,3,\cdots m-3 \end{cases}\\
&\text{end for}\\
&\text{For } i = 1,2,\cdots m-2 \text{ do :}\\
\ell((i,j)(i,j+1)) &= \begin{cases} a, & j = 0,2,\cdots n-3\\ b, & j = 1,3,\cdots n-2 \end{cases}\\
&\text{end for}
\end{aligned}
$$

Obviously $P_m \square P_n$ is a-sum V_4 magic.

Case 3: m is odd and n is even.

By interchanging the roles of m and n in Case 2, we get $\ell^+(v) = a$ for all $v \in V(P_m \Box P_n)$.

This completes the proof.

Theorem 2.31. *(see [5]) $P_m \Box P_n \in \mathscr{V}_0$ for all m and n.*

Theorem 2.32. *$P_m \Box P_n \in \mathscr{V}_{a,0}$ if and only if mn is even.*

Proof. The proof follows from theorems 2.30 and 2.31. □

Theorem 2.33. *For $m, n \geq 2$, the complete bipartite graph $K(m, n)$ is a-sum V_4 magic if and only if $m + n$ is even.*

Proof. First assume that $K(m, n)$ is a-sum V_4 magic. Let $\{u_i, i = 1, 2, \cdots m\} \cup \{v_j, j = 1, 2, \cdots n\}$ be the vertices of the graph with $E(G) = \{u_i v_j : i = 1, 2, \cdots m, j = 1, 2, \cdots n\}$. Then we have $\sum_{i=1}^{m} \ell^+(u_i) + \sum_{j=1}^{n} \ell^+(v_j) = 0$. This implies that $m + n$ is even. Conversely suppose that $m + n$ is even. Then we have the following cases. □

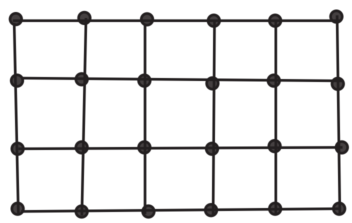

Fig. 5. Planar grid $P_6 \Box P_4$

Case 1: m and n are odd.

Define a labeling $\ell : E(K(m, n)) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} &\text{For } i = 1, 2, \cdots m \text{ do :} \\ &\ell(u_i v_j) = a, \ j = 1, 2, \cdots n. \\ &\text{end for} \end{aligned}$$

Thus $\ell^+(v) = a$.

Case 2: m and n are even.

$$\begin{aligned} &\ell(u_i v_2) = c, i = 1, 3, 4, \cdots m, \\ &\ell(u_2 v_j) = c, j = 1, 3, 4, \cdots n, \\ &\ell(u_2 v_2) = b, \\ &\text{For } i = 1, 3, 4, \cdots m \text{ do :} \\ &\ell(u_i v_j) = b, j = 1, 3, 4, \cdots n. \\ &\text{end for} \end{aligned}$$

Obviously, $K(m,n)$ is a-sum V_4 magic. □

Theorem 2.34. *(see [2]) $K(m,n)$ is zero-sum V_4 magic for all m and n.*

Theorem 2.35. *$K(m,n) \in \mathscr{V}_{a,0}$ if and only if $m+n$ is even.*

Proof. The proof follows from theorems 2.33 and 2.34. □

Definition 2.10. (see [2]) When k copies of C_n share a common edge it will form the n-gon book of k pages and is denoted by $B(n,k)$.

The graph $B(6,4)$ is shown in Fig. 6.

Lemma 2.36. *If ℓ is an a-sum V_4 magic labeling of $B(n,k)$, then*

$$\sum_{i=1}^{n-2}\sum_{j=1}^{k}\ell^+(u_i^j) + \ell^+(u) + \ell^+(v) = 0$$

where u and v are the common vertices and $u_1^j, u_2^j, \cdots u_{n-2}^j$ are the other vertices of the j^{th}page.

Proof. The proof is similar to Lemma 2.1. □

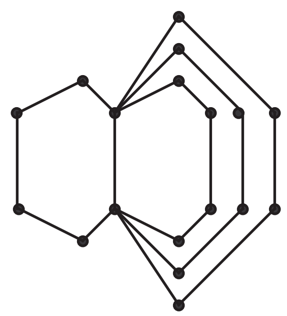

Fig. 6. The graph $B(6,4)$

Theorem 2.37. *For any $n \geq 3$ and $k \geq 1$, $B(n,k) \in V_a$ if and only if $(n-2)k$ is even.*

Proof. First assume that $B(n,k) \in V_a$. Then $[(n-2)k+2]a = 0$.This implies that $(n-2)k$ is even. Conversely assume that $(n-2)k$ is even. We consider the following cases.

Case 1: Both n and k are even.

Define a labeling $\ell : E(B(n,k)) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned}
\ell(uv) &= a, \\
\text{For } j = 1,2,\cdots k \text{ do :} \\
\ell(uu_1^j) &= \ell(vu_{(n-2)}^j) = b, \\
\ell(u_{2i}^j u_{2i+1}^j) &= b, \text{for } i = 1,2,\cdots \frac{n-4}{2}, \\
\ell(u_{2i-1}^j u_{2i}^j) &= c, \text{for } i = 1,2,\cdots \frac{n-4}{2}. \\
\text{end for}
\end{aligned}$$

Then ℓ is an a-sum V_4 magic labeling of $B(n,k)$.

Case: 2 n is odd and k is even,

Define a labeling $\ell : E(B(n,k)) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned}
\ell(uv) &= a, \\
\text{For } j = 1,2,\cdots k \text{ do :} \\
\ell(uu_1^j) &= b, \\
\ell(vu_{(n-2)}^j) &= c, \\
\ell(u_{2i}^j u_{2i+1}^j) &= b, \text{for } i = 1,2,\cdots \frac{n-3}{2}, \\
\ell(u_{2i-1}^j u_{2i}^j) &= c, \text{for } i = 1,2,\cdots \frac{n-3}{2}. \\
\text{end for}
\end{aligned}$$

Case 3: n is even and k is odd

Define a labeling $\ell : E(B(n,k)) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned}
\ell(uv) &= c, \\
\text{For } j = 1,2,\cdots k \text{ do :} \\
\ell(uu_1^j) &= \ell(vu_{(n-2)}^j) = b, \\
\ell(u_{2i}^j u_{2i+1}^j) &= b, \text{for } i = 1,2,\cdots \frac{n-4}{2}, \\
\ell(u_{2i-1}^j u_{2i}^j) &= c, \text{for } i = 1,2,\cdots \frac{n-2}{2}. \\
\text{end for}
\end{aligned}$$

Thus ℓ is an a-sum V_4 magic labeling of $B(n,k)$. □

Theorem 2.38. *(see [2]) For any $n \geq 3$ and $k \geq 1, B(n,k)$ is zero-sum V_4 magic.*

Theorem 2.39. *For any $n \geq 3$ and $k \geq 1, B(n,k) \in \mathscr{V}_{a,0}$ if and only if $(n-2)k$ is even.*

Proof. The proof follows from theorems 2.37 and 2.38. □

Definition 2.11. (see [5]) The book B_n is the graph $S_n \square P_2$ where S_n is the star with $n+1$ vertices.

The book graph B_4 is shown in Fig. 7.

Theorem 2.40. *The book B_n is a-sum V_4 magic for all n.*

Proof. Let w_1, w_2 be the vertices of the common edge. Let $\{u_1, u_2, \cdots u_n\} \cup \{v_1, v_2, \cdots v_n\}$ be the vertices of B_n.

Case 1: n is odd.

$$\begin{aligned} \ell(w_1w_2) &= c, \\ &\text{For } i = 1, 2, \cdots n \text{ do :} \\ \ell(w_1u_i) &= \ell(w_2v_i) = b, \\ \ell(u_iv_i) &= c. \\ &\text{end for} \end{aligned}$$

Case 2: n is even.

$$\begin{aligned} \ell(w_1w_2) &= a, \\ &\text{For } i = 1, 2, \cdots n \text{ do :} \\ \ell(w_1u_i) &= \ell(w_2v_i) = b, \\ \ell(u_iv_i) &= c. \\ &\text{end for} \end{aligned}$$

Clearly ℓ is an a-sum V_4 magic labeling of B_n. □

Theorem 2.41. *$B_n \in \mathscr{V}_0$ for all n.*

Proof. We consider the following cases:
Case 1: n is odd.

Label all the edges by a. Then we get $\ell^+(v) = 0$ for all $v \in V(G)$. Case 2: n is even.

Define a labeling $\ell : E(B_n) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned} \ell(w_1w_2) &= a, \\ \ell(u_1w_1) &= \ell(v_1w_1) = \ell(u_1v_1) = c, \\ \ell(w_1u_i) &= \ell(w_2v_i) = b, i = 2, 3, \cdots n, \\ \ell(u_iv_i) &= b, i = 2, 3, \cdots n. \end{aligned}$$

With this labeling we get $\ell^+(v) = 0$ for all $v \in V(G)$.

Definition 2.12. (see [2]) Given a cycle C_n construct a cycle C_m on each edge of this cycle. The resulting graph is called flower graph and is denoted by $C_m@C_n$.

Theorem 2.42. *For all $m, n \geq 3$, the flower graph $C_m@C_n \in \mathscr{V}_a$ if and only if $n(m-1)$ is even.*

Proof. Suppose that $C_m@C_n \in \mathscr{V}_a$. Then $[n(m-1)]a = 0$. This implies that $n(m-1)$ is even. Now let $u_1, u_2, \cdots u_n$ be the vertices of C_n and $v_{1j}, v_{2j}, \cdots v_{(m-2)j}$ be the vertices of $C_j, j = 1, 2, \cdots n$. We consider the following cases.

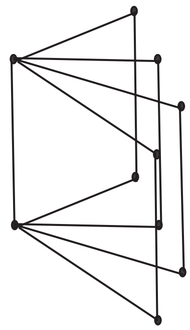

Fig. 7. Book graph B_4

Case 1: n is even and m is odd.

$$\begin{aligned} &\text{For } j = 1, 2, \cdots n \text{ do :}\\ \ell(u_j u_{(j+1)}) &= a,\\ \ell(u_j v_{1j}) &= c,\\ \ell(u_{(j+1)} v_{(m-2)j}) &= b,\\ \ell(v_{ij} v_{i(j+1)}) &= b,\ i = 1, 3, \cdots m-4,\\ \ell(v_{ij} v_{i(j+1)}) &= c,\ i = 2, 4, \cdots m-3.\\ &\text{end for} \end{aligned}$$

Case 2: Both n and m are even.

$$\begin{aligned} \ell(u_j u_{(j+1)}) &= a,\ j = 1, 2, \cdots n,\\ &\text{For } j = 1, 3, \cdots n-1 \text{ do :}\\ \ell(v_{ij} v_{i(j+1)}) &= c,\ i = 1, 3, \cdots m-3,\\ \ell(v_{ij} v_{i(j+1)}) &= b,\ i = 2, 4, \cdots m-4.\\ &\text{end for}\\ &\text{For } j = 2, 4, \cdots n \text{ do :}\\ \ell(v_{ij} v_{i(j+1)}) &= b,\ i = 1, 3, \cdots m-3,\\ \ell(v_{ij} v_{i(j+1)}) &= c,\ i = 2, 4, \cdots m-4.\\ &\text{end for} \end{aligned}$$

Case 3: Both n and m are odd.

$$\begin{aligned} &\text{For } j = 1, 2, \cdots n \text{ do :}\\ \ell(u_j u_{(j+1)}) &= a,\\ \ell(u_j v_{1j}) &= b,\\ \ell(u_{(j+1)} v_{(m-2)j}) &= c,\\ \ell(v_{ij} v_{i(j+1)}) &= c,\ i = 1, 3, \cdots m-4,\\ \ell(v_{ij} v_{i(j+1)}) &= b,\ i = 2, 4, \cdots m-3.\\ &\text{end for} \end{aligned}$$

Thus ℓ is an a-sum V_4 magic labeling of $C_m@C_n$. This completes the proof. □

Theorem 2.43. *(see [2]) For all $m, n \geq 3$, the flower graph $C_m@C_n \in \mathscr{V}_0$.*

Theorem 2.44. *For all $m, n \geq 3$, the flower graph $C_m@C_n \in \mathscr{V}_{a,0}$ if and only if $n(m-1)$ is even.*

Proof. The proof follows from theorems 2.42 and 2.43. □

Definition 2.13. (see [2]) Given k natural numbers $a_1, a_2, \cdots a_k$, if we connect the two vertices of $N_2 = \{u, v\}$ by k parallel paths of length $a_1, a_2, \cdots a_k$, the resulting graph is called the generalized Theta graph and is denoted by $\Theta(a_1, a_2, \cdots a_k)$.

Note that in this graph, deg(u)= deg(v)= k and all the other vertices are of degree two.

Theorem 2.45. *If the generalized Theta graph $\Theta(a_1, a_2, \cdots a_k)$ is a-sum V_4 magic then either odd number of a_i's are odd or even number of a_i's are even.*

Proof. First suppose that $\Theta(a_1, a_2, \cdots a_k)$ is a-sum V_4 magic. Then we have $\left[\sum_{i=1}^{k} a_i - k\right] a = 0$. This implies that $\left[\sum_{i=1}^{k} a_i\right] a = ka$. This is if and only if both $\sum_{i=1}^{k} a_i$ and k are odd or even simultaneously. This happens if and only if odd number of a_i's are odd or even number of a_i's are even. □

Theorem 2.46. *Let $\Theta(a_1, a_2, \cdots a_k)$ be a generalized Theta graph. If k and even number of a_i's are even then $\Theta(a_1, a_2, \cdots a_k)$ is a-sum V_4 magic.*

Proof. Let $v_1^i, v_2^i, \cdots v_{a_i-1}^i$ be the vertices of the i^{th} path and let u,w be the common vertices. Define a labeling $\ell : \Theta(a_1, a_2, \cdots a_k) \longrightarrow V_4 \setminus \{0\}$ by

$$\begin{aligned}
&\text{For } i = 1, 2, \cdots k-1 \text{ do :}\\
\ell(uv_1^i) &= b\\
&\text{end for}\\
\ell(uv_1^k) &= c\\
&\text{For } i = 1, 2, \cdots k \text{ do :}\\
\ell(v_j^i v_{j+1}^i) &= \begin{cases} c, & j = 1, 3, \cdots a_i - 2, \quad \text{if } a_i \text{ is odd}\\ & j = 1, 3, \cdots a_i - 3, \quad \text{if } a_i \text{ is even}\end{cases}\\
\ell(v_j^i v_{j+1}^i) &= \begin{cases} b, & j = 2, 4, \cdots a_i - 2, \quad \text{if } a_i \text{ is even}\\ & j = 2, 4, \cdots a_i - 3, \quad \text{if } a_i \text{ is odd}\end{cases}
\end{aligned}$$

Now label the edge $v_{a_k-1}^i w, i = 1, 2, \cdots k$ in the following way.

If $\ell(v_{a_k-2}^i v_{a_k-1}^i) = b$, let $\ell(v_{a_k-1}^i w) = c$ and viceversa. Thus ℓ is an a-sum V_4 magic labeling of $\Theta(a_1, a_2, \cdots a_k)$. □

Theorem 2.47. *(see [2])* $\Theta(a_1, a_2, \cdots a_k)$ *is zero-sum* V_4 *magic for any sequence* $a_1, a_2, \cdots a_k$.

Theorem 2.48. *Let* $\Theta(a_1, a_2, \cdots a_k)$ *be a generalized Theta graph. If* k *and even number of* a_i*'s are even then* $\Theta(a_1, a_2, \cdots a_k) \in \mathscr{V}_{a,0}$.

Proof. The proof follows from theorems 2.46 and 2.47. □

Definition 2.14. (see [2]) Let $N_2 = \{v_1, v_2\}$ be the disconnected graph of order two. The graph $C_n \vee N_2$ is called the bypyramid based on C_n and is denoted by $BP(n)$.

The bypyramid graph based on C_6 denoted as $BP(6)$ is shown in Fig. 8.

Theorem 2.49. *For any* $n \geq 4$, *the bipyramid graph* $BP(n)$ *is a-sum* V_4 *magic if and only if* n *is even.*

Proof. Suppose that $BP(n) \in \mathscr{V}_a$. This implies that $(n+2)a = 0$. Thus we have obtained n is even.Conversely assume that n is even.

$$\begin{aligned}
&\text{For } i = 1,2 \text{ do :}\\
\ell(v_i u_1) &= c,\\
\ell(v_i u_j) &= b,\ j = 2,3,\cdots n\\
&\text{end for}\\
\ell(u_j u_{(j+1)}) &= b,\ j = 1,3,\cdots n-1,\\
\ell(u_j u_{(j+1)}) &= c,\ j = 2,4,\cdots n.
\end{aligned}$$

Obviously $BP(n) \in \mathscr{V}_a$. □

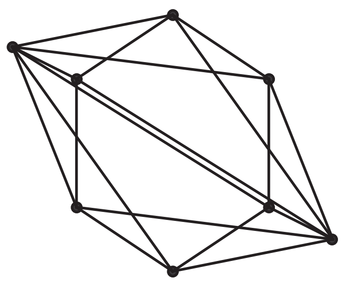

Fig. 8. The graph $BP(6)$

Theorem 2.50. *(see [2]) For any* $n \geq 4, BP(n)$ *is zero-sum* V_4 *magic for all* n.

Theorem 2.51. *For any* $n \geq 4, BP(n) \in \mathscr{V}_{a,0}$ *if and only if* n *is even.*

Proof. The proof follows from theorems 2.49 and 2.50. □

Definition 2.15. (see [2]) Given a graph G, we can define the bypyramid based on G to be $G \vee N_2$. This graph will be denoted by $BP(G)$.

Theorem 2.52. *Consider the bypyramid graph $BP(G)$ based on G. We have the following.*

i) If G is a-sum V_4 magic and number of vertices in G is odd, then $BP(G)$ is a-sum V_4 magic.

ii) If G is a-sum V_4 magic and number of vertices in G is even, then $BP(G)$ is 0-sum V_4 magic.

iii) If G is 0-sum V_4 magic and number of vertices in G is even, then $BP(G)$ is both a-sum V_4 magic and 0-sum V_4 magic.

iv) If G is 0-sum V_4 magic and number of vertices in G is odd, then $BP(G)$ is 0-sum V_4 magic.

Proof. Obvious. □

Corollary 2.53. *We have the following.*

i) If G is 0-sum V_4 magic, then $G \vee \overline{K_n} \in \mathscr{V}_{a,0}$.

ii) If G is a-sum V_4 magic, then $G \vee \overline{K_n} \in \mathscr{V}_{a,0}$.

3 Conclusions

This paper is a continuation of the work carried out in [3] and [4]. In this paper we investigated graphs in the following categories:

1) $\mathscr{V}_a$, the class of a-sum V_4 magic graphs.

2) $\mathscr{V}_0$, the class of zero-sum V_4 magic graphs.

3) $\mathscr{V}_{a,0}$, the class of graphs which are both a-sum and zero-sum V_4 magic.

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Competing Interests

The authors declare that no competing interests exist.

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