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Solving Differenced Kepler's Equation Using Homotopic Continuation Method

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

this paper deals with convergent ≥ 2 of arbitrary order through an effective iterative method, it is formulated to solve difference Kepler equation. This formulation with a dynamical aspect, where go from one iterative model to another one using more instruction. Where, the more important hint that, it is not need any prior information of initial guesses and keep away from the critical situations between divergent to the very slow convergent solutios, which may exist in another numerical method that depends on the initial guesses. Finally, copmuted algorithm and numerical example for the method are gevin.

Keywords: Initial value problem; differenced Kepler's equation; homotopy continuation method; space dynamics; mathematical astronomy.

1. INTRODUCTION

Most of applied mathematics resulting equations with high transcendental and can solve using iterative methods through (a) initial gusses (b) iterative model. Really, the last two points not separated from each other, so, agreement even exact iterative schemes are very cretical to the initial guess. More of that, the first guess might lead to extreme situation between divergent and

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very slow convergent solutions within much cases. "In a field of numerical analysis, the powerful techniques devoted to solve the transcendental equations without any prior information of the initial guess" [1], it is known as homotopy continuation method. It was applied firstly in space dynamics for the univeral initial value proplem [2] and also in stellar statistics [3] and also for the algebraic equation which nonlinear as more of researches as [4-7].

In the present work, we will establish an efficient iterative method of arbitrary integer order l of convergent l mor than or equal 2 to solve difference kepler equation. This hypothesis is of dynamical form, where, it goes from an iterative model to the next one, needed only more guidance. As further, the more significance of this hypothesis that, it is not need a prior information of initial guesses. A behavior which side step the critical cases between divergent solution to a very dawdling converge solutions, that may occure in another numerical methods which depend on initial estimate. Finally, we state a computaional algorithm of digital implementation for the hypothesis

2. A ONE-POINT ITERATION FORMULAE TO SOLVING Y(x) = 0

Put Y(x) = 0 where Y: R $\rightarrow$ R as a mapping with a solution $x = \eta(say)$. We will recall some basic definitions to construct an iterative model to solve this equation:

The error in k th iterate is defined as follows

$$\delta_k = \eta - x_k.$$

If the sequence $\{x_k\}$ is converges to $x = \eta,$ then

$$\lim_{k\to\infty} x_k = \eta$$

Suppose a real number $p \geq 1$ where:

$$\lim_{i\to\infty}\frac{|x_{i+1}-\eta|}{|x_i-\eta|^p} = \lim_{i\to\infty}\frac{|\delta_{i+1}|}{|\delta_i|^p} = K \neq 0$$

Can state , an iterated mode in order p at η and constant K is known as the converge error constant. If p =1 , then convergence is linear , at p = 2, then convergence is quadratic; and if p =3,4,5 find that the convergence is cubic, quartic and quintic, respectively.

Using information at only one point for one-point reputation formulae. So, considering only stable one-point improvement formulae as below:

$$x_{i+1} = R(x_i), \quad i = 0\,, \quad 1,\cdots \tag{1}$$

The order of one point iterative formulae can determine from: (a) The Taylor series of the iteration function R (x_n) about η e.g [8]. or from , (b) The Taylor series of the function $Y(x_{k+1})$ about x_k [9].

From the concepts of the second supposes as shown above [point(b)] easy to form a class of repeated formulae consists members of all integrating orders [10] for solving equation (1) as follows:

$$x_{i+1} = x_i + \sigma_{i,m+2}\text{; i=0,1,2,…; m =0,1,2,} \tag{2}$$

Since

$$\sigma_{i,m+2} = \frac{-Y_i}{\sum_{j=1}^{m+1}(\sigma_{i,m+1})^{j-1}Y_i^{(j)}/j!};$$
$$\sigma_{i,1} = 1; \forall i \geq 0. \tag{3}$$

$$\mathrm{Y}_\mathrm{i}^{(\mathrm{j})} \equiv \left.\frac{\mathrm{d}^\mathrm{j}\mathrm{Y}(\mathrm{x})}{\mathrm{dx}^\mathrm{j}}\right|_{\mathrm{x}=\mathrm{x}_\mathrm{i}}; \; Y_i \equiv Y_i^{(0)}. \tag{4}$$

The convergence order is $\mathrm{m}+2$ and given as follows:

$$\delta_{i+1} = -\frac{1}{(m+2)!}\,\frac{Y(\eta)^{(m+2)}}{Y_i^{(1)}(\iota_1)}\delta_i^{m+2}, \tag{5}$$

where η between $\mathrm{x}_{\mathrm{i+1}}$ and x_i also η_1 between $\mathrm{x}_{\mathrm{i+1}}$ and η.

3. SOLVING Y(x) = 0 BY A HOMOTOPIC CONTINUATION METHODS

3.1 Constructions

To find the solution of single non-linear equation in one variable as x (say).

$$Y(\,x\,) = 0\,, \tag{6}$$

Since , $\mathrm{Y}:\mathbf{R}\rightarrow\mathbf{R}$ is a mapping , assuming our application will be smooth, so that a map requires many contiuous derivatives. Look to the status, where no prior information concerning available zero point of Y. Where, assuming that a prior information is un- available, so often some iterative methods are fail to calculate zero $\overline{\mathrm{x}}$, because of pooring chosen the starting value as an available repair, homotopy or deformation define

$\mathrm{H}:\mathbf{R}\times\mathbf{R} \rightarrow \mathbf{R}$ as follwos:

$$\mathrm{H}(\mathrm{x},1)=\mathrm{Q}(\mathrm{x}) \quad ; \quad \mathrm{H}(\mathrm{x},0)=\mathrm{Y}(\mathrm{x}),$$

- Since $\mathrm{Q}:\mathbf{R}\rightarrow\mathbf{R}$ is a (measly) smooth mapping have a known zero point and H is also smooth. Usually, can choose a convex

$$\mathrm{H}(\mathrm{x},\lambda)=\lambda\ \mathrm{Q}(\mathrm{x})+(1-\lambda)\,\mathrm{Y}(\mathrm{x}). \qquad (7)$$

and endeavor to tracing implicitly defined curve $\Phi(\mathrm{z})\in\mathrm{H}^{-1}(0)$ from the starting point $(\mathrm{x}_1,1)$ to the solution point $(\overline{\mathrm{x}},0)$. If it is succeed, the zero point $\overline{\mathrm{x}}$ of Y is obtained.

3.2 Embedding Methods

The embedding method which referred at the end of Subsection 3-1 is explained in the following algorithm for tracing the curve $\Phi(\mathrm{z})\in\mathrm{H}^{-1}(0)$ from, say $\lambda=1$ to $\lambda=0$.

3.2.1 Computational algorithm1

- **Purpose**

To solve Y(x) = 0 by embedding method.

- **Input**

(1) The function Q(x) with defined root x_1 such that H (x_1,1) = 0,
(2) Positive integer m.

- **Output**

Solution x of Y(x) = 0.

3.2.2 Computational sequence

Let $x=x_1,\ \lambda=(m-1)/m,\ \Delta\lambda=1/m.$

For i = 1 to m do begin

Solve H (y, $\lambda)=\lambda\mathrm{Q}(\mathrm{y})+(1-\lambda)\mathrm{Y}(\mathrm{y})=0$ iteratively for y using x as starting value.

x = y.

$\lambda=\lambda-\Delta\lambda.$

end

4. SOLVING DIFFERENCED KEPLER'S EQUATION USING HOMOTOPY METHOD'S APPLICATION

4.1 Formulated of Differenced Kepler's Equation

Since $(M_n,M_\ell),(E_n,E_\ell)$ the mean and eccentric anomalies are associated with position vectors $(\mathbf{r_n},\mathbf{r}_\ell)$ at the two epochs $\mathrm{t_n}$, t_ℓ where $(\ell\succ n)$ of an elliptic orbit.

Then, differenced Kepler's equation given as:

$$(M_\ell-M_n)=(E_\ell-E_n) \quad +\frac{\sigma n}{\sqrt{a}} \quad \{1-\cos(E_\ell-E_n)\}-\left(1-\frac{r_n}{a}\right)\sin(E_\ell-E_n),$$

Where

$$\sigma_\mathrm{n}=\frac{1}{\sqrt{\mu}}<\mathbf{r_n},\mathbf{v_n}>$$

The last equation has written as:

$$\mathrm{Y}=\mathrm{G}-\mathrm{C_n}\sin\mathrm{G}-\mathrm{S_n}\cos\mathrm{G}+\mathrm{S_n}-\mathrm{W}=0, \qquad (8)$$

Where

$$\mathrm{W}=\mathrm{M}_\ell-\mathrm{M_n}\ ,\ \mathrm{G}=\mathrm{E}_\ell-\mathrm{E_n}\ ;$$

$$\mathrm{C_n}=1-\frac{\mathrm{r_n}}{\mathrm{a}}\ ;\ \mathrm{S_n}=\frac{\sigma_\mathrm{n}}{\sqrt{\mathrm{a}}}. \qquad (9)$$

By checking the computed value G under the next condition:

$$\mathrm{Check}=\mathrm{G}-\mathrm{C_n}\sin\mathrm{G}-\mathrm{S_n}\cos\mathrm{G}+\mathrm{S_n}-\mathrm{W}\approx 0$$

We recoded two:

Obviously from equation (5), the iterative planner to solve equation (8) including the derivations of Y as higher as the order of model. In reality, the higher of accuracy and rate of convergence resulting from the higher order of the iterative model. So that the wonderful simplifying of derivatives formula of Y as follows:

$$Y^{(1)}(G) = 1 - C_n \cos G + S_n \sin G,$$

$$Y^{(2)}(G) = C_n \sin G - S_n \cos G,$$

$$Y^{(3)}(G) = C_n \cos G - S_n \sin G,$$

$$Y^{(k)}(G) = -Y^{(k-2)} \quad ; k \geq 4$$

So, can get all possible derivatives of Y (G).

Without prior information of initial guesses, homotopic continuation method is a sufficient technique for solving Y (G) = 0

By the last two hints, the solution of Kepler's Equation (8) can be found also an iterative algorithm of any positive integer with order l ≥2. Where, the algorithm unneeded a prior information about the initial guesses. Belonging to Equation (5), we only need to additional instruction for the algorithm of dynamical nature sense such that iterative schemes up to the l^{th} order which going from one scheme to the next one.

The algorithm is illustrated in the following with algorithm 1 which augmented it together with the Q function of the homotopy H , Equation (7), as Q(x) = x - 1, therefor

$$H(x_1, 1) = 0, \text{ where } x_1 = 1.$$

4.2 Computational Algorithm

Purpose: For solving kepler equation by using an iterative scheme of quadratic up to the l^{th} convergence orders without any prior information of initial guesses using the homotopy continuation method with $Q(G) = G - 1$

Input: m is positive integer where $3 \leq m \leq 20$, W , $Cn \ (\equiv C_n)$, $Sn (\equiv S_n)$, n $(\equiv \ell)$, ε (specific toleration $\approx 10^{-6}$) ,

4.3 Computational Sequence

Put $G = 1$; $\Delta\lambda = 1/m$; $\lambda = 1 - \Delta\lambda$

For i: =1 to m do

Start {i}

$$Q = 1 - \lambda$$

$$Y = \lambda \ (G-1) + Q \ \{G - Cn \ \sin G - Sn \cos G + Sn - W\}$$

$$Y^{(1)} = \lambda + Q\{1 - Cn \ \cos G + Sn \ Sin\, G\}$$

$$\Delta G = -\frac{Y}{Y^{(1)}}$$

If $[\ell = \cdots 2, \quad If \quad [\,|\Delta G| \leq \in,$ go to step 4]

$$Y^{(2)} = Q\,(Cn \sin G + Sn \cos G)$$

$$H = Y^{(1)} + D\,E * \frac{Y^{(2)}}{2}$$

$$\Delta G = -\frac{Y}{H}$$

If $[\ell = 3, \quad If \quad [\,|\Delta G| \leq \in,$ go to step 4]

$$Y^{(3)} = Q\,(Cn \cos G - Sn \sin G)$$

$$H = Y^{(1)} + DE * Y^{(2)}/2 + (DE)^2 * Y^{(3)}/6$$

$$\Delta G = -Y/H$$

If $[\ell = 4, If [|\Delta G| \leq \in,$ go to step 4]

$$L = \ell - 1$$

For k: = 4 to L do

begin { k }

put $Y^{(k)} = -Y^{(k-2)};$ $n = k - 1;$

$H = Y^{(1)} \ ; B = 1$

For j: = 1 to n do

begin { j }

$$B = \Delta G \ * \frac{B}{(j+1)}$$

$$H = H + B * Y^{(j+1)}$$

end { j }

$$\Delta G = -\frac{Y}{H}$$

end { k }

$$G = G + \Delta G$$

$$\lambda = \lambda - \Delta \lambda$$

end { i }

End

4.4 Numerical Example

Consider the values $W = 6.30025$; $C_n = -0.324852$; $S_n = 0.41876$ with $m = 10, \ell = 15, \varepsilon = 10.^{-6}$ .The result is: $G = 6.29604^r, \text{Check} = -8.88178 \times 10^{-16}$.

5. CONCLUSION

We will determined an effective iterative method of convergent arbitrary integer order l ≥ 2 for solving difference kepler equation. This effective method as the sense of dynamical problem which going from one iterative model to the next one, only needed an additional instruction. Additionally, the most important, that the procedure did not need a prior information of initial guesses. A property which keep away from the critical occurrences between divergent solutios to a very slow convergent, that may exist in another numerical method which depends on initial guesses. Lastly, we get a numerical package for digital execution of the hypothesis.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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