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# Completion of BCC-algebras

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**Abstract:** In this paper, we study some properties of induced topology by a uniform space generated by a family of ideals of a BCC-algebra. Also, by using Cauchy nets we construct a uniform space which is completion of this space.

**Keywords:** *BCC*-algebra, uniform space, cauchy net, ideal.

**MSC:** 06B10, 03G10.

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## 1. Introduction

In 1966, Y. Imai and K. Iséki in [1] introduced a class of algebras of type $(2,0)$ called BCK-algebras which generalizes on one hand the notion of algebra of sets whit the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of implication algebra. K. Iséki posed an interesting problem whether the class of BCK-algebras form a variety. In connection with this problem Y. Komori in [2] introduced a notion of BCC-algebras which is a generalization of notion BCK-algebras and proved that class of all BCC-algebras is not a variety. W. A. Dudek in [3] redefined the notion of BCC-algebras by using a dual form of the ordinary definition. Further study of BCC-algebras was continued [4–6].

In 1937, André Weil in [7] introduced the concept of a uniform space as a generalization of the concept of a metric space in which many non-topological invariants can be defined. The study of quasi uniformities started in 1948 with Nachbin's investigations on uniform preordered spaces. Mehrshad and Kouhestani in [8] introduced a quasi-uniformity on a BCC-algebra by a family of ideals and studied some properties of this structure. Now, in this present work, we consider the set $C$ of all cauchy nets on BCC-algebras $X$ and define a congruence relation $\sim$ on this set. Then we consider the quotient BCC-algebra $\mathcal{C} = \frac{C}{\sim}$ and prove that $\mathcal{C}$ is a BCC-algebra. We construct a uniformity on $\mathcal{C}$ and show that this uniformity is a completion of uniform space on $X$ induced by a family of ideals of $X$.

## 2. Preliminary

### BCC-algebras

A BCC-algebra is a non empty set $X$ with a constant 0 and a binary operation $*$ satisfying the following axioms, for all $x, y, z \in X$ :

(1) $((x * y) * (z * y)) * (x * z) = 0$,
(2) $0 * x = 0$,
(3) $x * 0 = x$
(4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

A non empty subset $S$ of BCC-algebra $X$ is called subalgebra of $X$ if it is closed under BCC-operation. For a BCC-algebra $X$, we denote $x \wedge y = y * (y * x)$ for all $x, y \in X$. On any BCC-algebra $X$ one can define the natural order $\leq$ putting

$$x \leq y \Leftrightarrow x * y = 0.$$

It is not difficult to verify that this order is partial and 0 is its smallest element. In BCC-algebra $X$, following hold: for any $x, y, z \in X$

(5) $(x * y) * (z * y) \leq x * z$,
(6) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$,

(7) $x \wedge y \leq x, y$
(8) $x * y \leq x$
(9) $(x * y) * z \leq x * (y * z)$
(10) $x * x = 0$,
(11) $(x * y) * x = 0$ [ see, [6]].

**Definition 1.** [9] Let $X$ be a BCC-algebra and $\emptyset \neq I \subseteq X$. $I$ is called an ideal of $X$ if it satisfies the following conditions:

(12) $0 \in I$,
(13) $x * y \in I$ and $y \in I$ imply $x \in I$.

If $I$ is an ideal in BCC-algebra of $X$, then $I$ is a subalgebra. Moreover, if $x \in I$ and $y \leq x$, then $y \in I$. An ideal $I$ is said to be *regular ideal* if the relation

$$x \equiv^I y \Longleftrightarrow x * y, y * x \in I$$

is a congruence relation. In this case we denote $x/I = \{y : x \equiv^I y\}$ and $X/I = \{x/I : x \in X\}$. $X/I$ is a BCC-algebra by $x/I * y/I = (x * y)/I$.

**Uniform and quasi uniform space**

Let $A$ be a non-empty set and $\emptyset \neq \mathcal{F} \subseteq P(A)$. Then $\mathcal{F}$ is called a $filter$ on $P(A)$, if for each $F_1, F_2 \in \mathcal{F}$ :

(i) $F_1 \in \mathcal{F}$ and $F_1 \subseteq F$ imply $F \in \mathcal{F}$,
(ii) $F_1 \cap F_2 \in \mathcal{F}$,
(iii) $\emptyset \notin \mathcal{F}$.

A subset $\mathcal{B}$ of a filter $\mathcal{F}$ on $A$ is a *base* of $\mathcal{F}$ iff, every set of $\mathcal{F}$ contains a set of $\mathcal{B}$. If $\mathcal{F}$ is a family of nonempty subsets of $A$, then we denote generated filter by $\mathcal{F}$ with $fil(\mathcal{F})$.

A $quasi\text{-}uniformity$ on a set $A$ is a filter $Q$ on $P(X \times X)$ such that

(i) $\triangle = \{(x, x) \in A \times A : x \in A\} \subseteq q$, for each $q \in Q$,
(ii) For each $q \in Q$, there is a $p \in Q$ such that $p \circ p \subseteq q$ where

$$p \circ p = \{(x, y) \in A \times A : \exists z \in A \ s.t \ (x, z), (z, y) \in p\}.$$

The pair $(A, Q)$ is called a $quasi\text{-}uniform\ space$. If $Q$ is a quasi-uniformity on a set $A$, then $q^{-1} = \{q^{-1} : q \in Q\}$ is also a quasi-uniformity on $A$ called the $conjugate$ of $Q$. It is well-known that if a quasi-uniformity satisfies condition: $q \in Q$ implies $q^{-1} \in Q$, then $Q$ is a $uniformity$. Also $Q$ is a uniformity on $A$ provided

$$\forall q \in Q \ \exists p \in Q \ s.t \ p^{-1} \circ p \subseteq q.$$

Furthermore, $Q^* = Q \vee Q^{-1}$ is a uniformity on $A$. A subfamily $\mathcal{C}$ of quasi-uniformity $Q$ is said to be a base for $Q$ iff, each $q \in Q$ contains some member of $\mathcal{C}$. The topology $T(Q) = \{G \subseteq X : \forall x \in G \ \exists q \in Q \ s.t \ q(x) \subseteq G\}$ is called the topology induced by the quasi-uniformity $Q$ [ See, [10]].

## 3. Main results

Let $X$ be a $BCC$-algebra and $\eta$ be an arbitrary family of ideals of $X$ which is closed under intersection.

**Theorem 1.** *[8] Let $X$ be a BCC-algebra. The set $\mathcal{I} = \{I_L : I \in \eta\}$ is a base for a quasi uniformity $\mathcal{U}$ on $X$, where $I_L = \{(x, y) \in X \times X : y * x \in I\}$.*

**Lemma 1.** *[8] Let $I$ be a regular ideal of BCC- algebra $X$. Define $I_L^{-1} = \{(x, y) \in X \times X : (y, x) \in I_L\}$ and $I_L^{\star} = I_L \cap I_L^{-1}$. Then following holds:*

*(i)* $I_L^{-1} = \{(x, y) \in X \times X : x * y \in I\}$,
*(ii)* $I_L^{-1}(x) = \{y \in X : x * y \in I\}$,
*(iii)* $I_L^{-1}(0) = X$,
*(iv)* $I_L^{\star} = \{(x, y) \in X \times X : x \equiv^I y\}$,

*(v)* $I_L^\star(x) = \{y \in X : x \equiv^I y\} = x/I$,
*(vi)* *if* $x \in I$, *then* $I_L^\star(x) = I$.

**Theorem 2.** *[8] Let* $\mathcal{U}^\star = \{U \subseteq X \times X : \exists I \in \eta \;\; I_L^\star \subseteq U\}$. *Then the pair* $(X, \mathcal{U}^\star)$ *is a uniform space. Moreover,* $(X, T(\mathcal{U}^\star))$ *is a topological BCC-algebra, where* $T(\mathcal{U}^\star) = \{G \subseteq X : \forall x \in G \, \exists I \in \eta \;\; I_L^\star(x) \subseteq G\}$ *is the induced topology by* $\mathcal{U}^\star$ *on* $X$.

Let $J = \bigcap_{I \in \eta} I$. Then $\mathcal{U}^\star = \{U \subseteq X \times X : J_L^\star \subseteq U\}$ and $\tau_J = \{G \subseteq X : \forall x \in G \;\; J_L^\star(x) \subseteq G\}$.

**Proposition 1.** $T(\mathcal{U}^\star) = \tau_J$, *where* $J = \bigcap_{I \in \eta} I$.

**Proof.** Let $x \in G \in T(\mathcal{U}^\star)$. Then there exists $I \in \eta$ such that $I_L^\star(x) \subseteq G$. Since for any $I \in \eta$ $J \subseteq I$, we get $J_L^\star \subseteq I_L^\star$. Hence $J_L^\star(x) \subseteq I_L^\star(x) \subseteq G$ and so $G \in \tau_J$. Thus $T(\mathcal{U}^\star) \subseteq \tau_J$. Conversely, let $x \in G \in \tau_J$. Then $J_L^\star(x) \subseteq G$. Since $\eta$ is closed under intersection, $J \in \eta$ and so $J_L^\star \in \mathcal{U}^\star$. Hence $G \in T(\mathcal{U}^\star)$. Therefore $\tau_J \subseteq T(\mathcal{U}^\star)$. □

**Definition 2.** [11]

(i) A poset $(D, \leq)$ is called an *upward* directed set if for any $i, j \in D$ there exists $k \in D$ such that $i \leq k$ and $j \leq k$.
(ii) Let $(D, \leq)$ be an upward directed set and $X$ be a BCC-algebra. The mapping $x : D \to X$ is called a net in $X$ and denoted by $\{x_i\}_{i \in D}$.

**Definition 3.** Let $\{x_i\}_{i \in D}$ be a net in topological space $(X, \tau_J)$. Then

(i) $\{x_i\}_{i \in D}$ is called *converges* to $x \in X$ if for any neighborhood $G$ of $x$ there exists $i_0 \in D$ such that $x_i \in G$ for any $i \geq i_0$. In this case we write $x_i \to x$.
(ii) $\{x_i\}_{i \in D}$ is called *Cauchy* if there exists $i_0 \in D$ such that $\frac{x_i}{J} = \frac{x_j}{J}$ for any $i, j \geq i_0$.

**Proposition 2.** *Let* $\{x_i\}_{i \in D}$ *and* $\{y_i\}_{i \in D}$ *be two nets in* $(X, \tau_J)$. *Then*

*(i) If* $x, y \in X$, $x_i \to x$ *and* $y_i \to y$, *then* $x_i * y_i \to x * y$.
*(ii) Each convergent net in* $X$ *is a cauchy net.*

**Proof.** (i) Let $x * y \in G \in \tau_J$. Then $J_L^\star(x * y) \subseteq G$. Since $x_i \to x$ and $J_L^\star(x)$ is a neighbohood of $x$, there exists $i_0 \in D$ such that $x_i \in J_L^\star(x)$ for any $i \geq i_0$. Similarly, there exists $i_1 \in D$ such that $y_i \in I_L^\star(y)$ for any $i \geq i_1$. Since $D$ is an upward directed set, there exists $i_2 \in D$ such that $i_0, i_1 \leq i_2$. Hence by Lemma (1) $x_i * y_i \in J_L^\star(x) * J_L^\star(y) = \frac{x}{J} * \frac{y}{J} = \frac{x*y}{J} = J_L^\star(x * y) \subseteq G$ for any $i \geq i_2$ and so $x_i * y_i \to x * y$.
(ii) Let $\{x_i\}_{i \in D}$ be a net in $X$ and $x_i \to x \in X$. Since $J_L^\star(x)$ is a neighborhood of $x$, there exists $i_0 \in D$ such that $x_i \in J_L^\star(x)$ for any $i \geq i_0$. Hence $x_i \equiv^J x$ and $x_j \equiv^J x$ for any $i, j \geq i_0$ and so $x_i \equiv^J x_j$ for any $i, j \geq i_0$. Therefore $\frac{x_i}{J} = \frac{x_j}{J}$ for any $i, j \geq i_0$. Thus $\{x_i\}_{i \in D}$ is a cauchy net in $X$.
□

**Definition 4.** [11] Let $(A, Q)$ be a uniform space.

(i) A net $\{x_i\}_{i \in D}$ in $A$ is said to converge to a point $x \in A$ if for each $q \in Q$ there exists $i_0 \in D$ such that $(x_i, x) \in q$ for any $i \geq i_0$.
(ii) A net $\{x_i\}_{i \in D}$ in $A$ is said to be a Cauchy net if for each $q \in Q$ there exists $i_0 \in D$ such that $(x_i, x_j) \in q$ for any $i, j \geq i_0$.

Let $C$ be the set of all Cauchy sequence in $(X, \mathcal{U}^\star)$. define a binary relation on $C$ in the following way. For each $\{x_i\}_{i \in D}, \{y_j\}_{j \in D} \in C$, $\{x_i\}_{i \in D} \sim \{y_j\}_{j \in D}$ if and only if for all $U \in \mathcal{U}^\star$ there exist $i_0, j_0 \in D$ such that $(x_i, y_j) \in G$ for any $i \geq i_0$ and $j \geq j_0$.

**Theorem 3.** *The relation* $\sim$ *is a congruence relation on* $C$.

**Proof.** Since $(X, \mathcal{U}^\star)$ is a uniform space, $\triangle \subseteq U$ for any $U \in \mathcal{U}^\star$. Hence $(x_i, x_i) \in U$ for any $i \in D$ and so $\{x_i\}_{i \in D} \sim \{x_i\}_{i \in D}$. Let $\{x_i\}_{i \in D} \sim \{y_j\}_{j \in D}$. Then for all $U \in \mathcal{U}^\star$ there exist $i_0, j_0 \in D$ such that $(x_i, y_j) \in U$ for any $i \geq i_0$ and $j \geq j_0$. Since $U \in \mathcal{U}^\star$, $U^{-1} \in \mathcal{U}^\star$. By definition of $U^{-1}$ we have $(y_j, x_i) \in U^{-1}$ for any $i \geq i_0$

and $j \geq j_0$. Hence $\{y_j\}_{j\in D} \sim \{x_i\}_{i\in D}$. Let $\{x_i\}_{i\in D} \sim \{y_j\}_{j\in D}$ and $\{y_j\}_{j\in D} \sim \{z_i\}_{i\in D}$. Let $U \in \mathcal{U}^\star$. There exists $V \in \mathcal{U}^\star$ such that $V \circ V \subseteq U$. Since $\{x_i\}_{i\in D} \sim \{y_j\}_{j\in D}$, there exist $i_0, j_0 \in D$ such that $(x_i, y_j) \in V$ for any $i \geq i_0, j \geq j_0$. Similarly, there exist $k_0, l_0 \in D$ such that $(y_j, z_k) \in V$ for any $j \geq l_0, k \geq k_0$. Since $D$ is an upward directed set, there exsits $n \in D$ such that $j_0, l_0 \leq n$. If $j \geq n$, then $(x_i, y_j) \in V$ and $(y_j, z_k) \in V$ for any $i \geq i_0$ and $k \geq k_0$. Hence $(x_i, z_k) \in V \circ V \subseteq U$ for any $i \geq i_0$ and $k \geq k_0$ and so $\{x_i\}_{i\in D} \sim \{z_k\}_{k\in D}$. Thus $\sim$ is an equivalence relation on $C$. Finally, we show that $\sim$ is congruence. Let $I \in \eta$, $\{x_i\}_{i\in D} \sim \{y_j\}_{j\in D}$ and $\{z_k\}_{k\in D} \sim \{w_l\}_{l\in D}$. Hence there exist $i_0, j_0, k_0$ and $l_0 \in D$ such that $(x_i, y_j) \in I_L^\star$ for any $i \geq i_0, j \geq j_0$ and $(z_k, w_l) \in I_L^\star$ for any $k \geq k_0$ and $l \geq i_0$. Let $i \geq i_0, j \geq j_0$ and $k \geq k_0$. Then $y_j \in I_L^\star(x_i)$ and $z_k \in L_L^\star(z_k)$. Thus $y_j * z_k \in I_L(x_i) * z_k \subseteq I_L^\star(x_i) * I_L^\star(z_k) = I_L^\star(x_i * z_k)$ and so $(x_i * z_k, y_j * z_k) \in I_L^\star$. Similarly, If $j \geq j_0, k \geq k_0$ and $l \geq l_0$, then $(y_j * z_k, y_j * w_l) \in I_L^\star$. Thus $(x_i * y_j, z_k * w_l) \in I_L^\star \circ I_L^\star subseteq I_L^\star$ for any $i \geq i_0, j \geq j_0, k \geq k_0$ and $l \geq l_0$. Since for each $U \in \mathcal{U}^\star$ there exists $I \in \eta$ such that $I_L^\star \subseteq U$, $(x_i * y_j, z_k * w_l) \in U$ for $i \geq i_0, j \geq j_0, k \geq k_0$ and $l \geq l_0$. Hence $\sim$ is a congruence relation on $C$. □

Let $\mathcal{C} = \frac{C}{\sim}$. Define a binary operation on $\mathcal{C}$ as follow:

$$* : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \quad \left( \frac{\{x_i\}_{i\in D}}{\sim}, \frac{\{y_j\}_{j\in D}}{\sim} \right) \to \frac{\{x_i * y_j\}_{i,j\in D}}{\sim}.$$

**Theorem 4.** $\left(\mathcal{C}, *, \frac{\{0\}_{i\in D}}{\sim}\right)$ *is a BCC-algebra.*

**Proof.** The proof is clear. □

Let $\mathcal{V} = \{\hat{U} : U \in \mathcal{U}^\star\}$ where,

$$\hat{U} = \left\{ \left( \frac{\{x_i\}_{i\in D}}{\sim}, \frac{\{y_j\}_{j\in D}}{\sim} \right) \in \mathcal{C} \times \mathcal{C} : \exists i_0, j_0 \in D : \ \forall i \geq i_0, j \geq j_0, \ (x_i, y_j) \in U \right\}.$$

**Theorem 5.** *The pair* $(\mathcal{C}, \mathcal{V})$ *is a uniform space.*

**Proof.** Let $\hat{U} \in \mathcal{V}$ and $\frac{\{x_i\}_{i\in D}}{\sim} \in \mathcal{C}$. Since $\{x_i\}_{i\in D} \sim \{x_i\}_{i\in D}$, there exists $i_0 \in D$ such that $(x_i, x_i) \in U$ for any $i \geq i_0$. Hence $\left(\frac{\{x_i\}_{i\in D}}{\sim}, \frac{\{x_i\}_{i\in D}}{\sim}\right) \in \hat{U}$. Since $\frac{\{x_i\}_{i\in D}}{\sim} \in \mathcal{C}$ is arbitrary, we get $\triangle \subseteq \hat{U}$. Let $\hat{U} \in \mathcal{V}$. Then $U \in \mathcal{U}^\star$ and so $U^{-1} \in \mathcal{U}^\star$. Hence $\widehat{U^{-1}} \in \mathcal{V}$. We show that $\widehat{U^{-1}} = (\hat{U})^{-1}$. Let $\left(\frac{\{x_i\}_{i\in D}}{\sim}, \frac{\{y_j\}_{j\in D}}{\sim}\right) \in (\hat{U})^{-1}$. Then $\left(\frac{\{y_j\}_{j\in D}}{\sim}, \frac{\{x_i\}_{j\in D}}{\sim}\right) \in \hat{U}$. Hence there exist $i_0, j_0 \in D$ such that $(y_j, x_i) \in U$ for any $i \geq i_0$ and $j \geq j_0$ and so $(x_i, y_j) \in U^{-1}$ for any $i \geq i_0$ and $j \geq j_0$. Therefore $\left(\frac{\{x_i\}_{i\in D}}{\sim}, \frac{\{y_j\}_{j\in D}}{\sim}\right) \in \widehat{U^{-1}}$ and hence $(\hat{U})^{-1} \subseteq \widehat{U^{-1}}$. Similarly, we have $\widehat{U^{-1}} \subseteq (\hat{U})^{-1}$. Thus $(\hat{U})^{-1} \in \mathcal{V}$ for any $\hat{U} \in \mathcal{V}$. Let $\hat{U} \in \mathcal{V}$. Then $U \in \mathcal{U}^\star$. There exists $V \in \mathcal{U}^\star$ such that $V \circ V \in U$. We claim that $\hat{V} \circ \hat{V} \subseteq \hat{U}$. Let $\left(\frac{\{x_i\}_{i\in D}}{\sim}, \frac{\{z_k\}_{k\in D}}{\sim}\right) \in \hat{V} \circ \hat{V}$. There exists $\frac{\{x_i\}_{i\in D}}{\sim} \in \mathcal{C}$ such that $\left(\frac{\{x_i\}_{i\in D}}{\sim}, \frac{\{y_j\}_{j\in D}}{\sim}\right) \in \hat{V}$ and $\left(\frac{\{y_j\}_{j\in D}}{\sim}, \frac{\{z_k\}_{k\in D}}{\sim}\right) \in \hat{V}$. Hence there exist $i_0, j_0, k_0$ and $l_0 \in D$ such that $(x_i, y_j) \in V$ for any $i \geq i_0, j \geq j_0$ and $(y_j, z_k) \in V$ for any $j \geq l_0, k \geq k_0$. Since $D$ is an upward direcred set, there exists $n \in D$ such that $n \geq j_0, l_0$. If $j \geq n$, then $(x_i, y_j) \in V$ and $(y_j, z_k) \in V$ for any $i \geq i_0, k \geq k_0$. Hence $(x_i, z_k) \in V \circ V \subseteq U$ for any $i \geq i_0, k \geq k_0$ and so $\left(\frac{\{x_i\}_{i\in D}}{\sim}, \frac{\{z_k\}_{k\in D}}{\sim}\right) \in \hat{U}$. Let $\hat{U}, \hat{V} \in \mathcal{V}$. Then $U, V \in \mathcal{U}^\star$ and so $U\cap \in V \in \mathcal{U}^\star$. Hence $\widehat{U \cap V} \in \mathcal{V}$. We show that $\widehat{U \cap V} = \hat{U} \cap \hat{V}$. Let $\left(\frac{\{x_i\}_{i\in D}}{\sim}, \frac{\{y_j\}_{j\in D}}{\sim}\right) \in \hat{U} \cap \hat{V}$. Then $\left(\frac{\{x_i\}_{i\in D}}{\sim}, \frac{\{y_j\}_{j\in D}}{\sim}\right) \in \hat{U}$ and $\left(\frac{\{x_i\}_{i\in D}}{\sim}, \frac{\{y_j\}_{j\in D}}{\sim}\right) \in \hat{V}$. There exist $i_0, j_0, i_1$ and $j_1 \in D$ such that $(x_i, y_j) \in U$ for any $i \geq i_0, j \geq j_0$ and $(x_i, y_j) \in V$ for any $i \geq i_1, j \geq j_1$. There exist $i_2, j_2 \in D$ such that $i_0, i_1 \leq i_2$ and $j_0, j_1 \leq j_2$. Hence $(x_i, y_j) \in U$ and $(x_i, y_j) \in V$ for any $i \geq i_2$ and $j \geq j_2$ and so $(x_i, y_j) \in U \cap V$ for any $i \geq i_2$ and $j \geq j_2$. Hence $\left(\frac{\{x_i\}_{i\in D}}{\sim}, \frac{\{y_j\}_{j\in D}}{\sim}\right) \in \widehat{U \cap V}$ and so $\hat{U} \cap \hat{V} \subseteq \widehat{U \cap V}$. Similarly, we can show that $\widehat{U \cap V} \subseteq \hat{U} \cap \hat{V}$. Finally,

let $\hat{U} \in \mathcal{V}$ and $\hat{U} \subseteq \widetilde{V} \subseteq \mathcal{C} \times \mathcal{C}$. We have to show that $\widetilde{V} \in \mathcal{V}$. Let $(x, y) \in U \in \mathcal{U}^{\star}$. Then $\left(\frac{\{x\}_{i\in D}}{\sim}, \frac{\{y\}_{j\in D}}{\sim}\right) \in \hat{U}$ and so $\left(\frac{\{x\}_{i\in D}}{\sim}, \frac{\{y\}_{j\in D}}{\sim}\right) \in \widetilde{V}$. Thus $(x, y) \in V$ and so $U \subseteq V$. Hence $V \in \mathcal{U}^{\star}$ and so $\widetilde{V} \in \mathcal{V}$.
□

**Theorem 6.** *$(\mathcal{C}, *, T(\mathcal{V}))$ is a topological BCC-algebra where,*

$$T(\mathcal{V}) = \left\{G \in \mathcal{C} : \forall\ \frac{\{x\}_{i\in D}}{\sim}\ \exists \hat{U} \in \mathcal{V}\ s.t.\ \hat{U}\left(\frac{\{x\}_{i\in D}}{\sim}\right) \subseteq G\right\}.$$

**Proof.** Let $\frac{\{x_i\}_{i\in D}}{\sim} * \frac{\{y_j\}_{j\in D}}{\sim} \in G \in T(\mathcal{V})$. Then there exists $U \in \mathcal{U}^{\star}$ such that $\hat{U}\left(\frac{\{x_i*y_j\}_{i,j\in D}}{\sim}\right) \subseteq G$. Since $U \in \mathcal{U}^{\star}$, there exists $I \in \eta$ such that $I_L^{\star} \subseteq U$. Clearly, $\widehat{I_L^{\star}}\left(\frac{\{x_i*y_j\}_{i,j\in D}}{\sim}\right) \subseteq \hat{U}\left(\frac{\{x_i*y_j\}_{i,j\in D}}{\sim}.\right)$ We claim that $\widehat{I_L^{\star}}\left(\frac{\{x_i\}_{i\in D}}{\sim}\right) * \widehat{I_L^{\star}}\left(\frac{\{y_j\}_{j\in D}}{\sim}\right) \subseteq \widehat{I_L^{\star}}\left(\frac{\{x_i*y_j\}_{i,j\in D}}{\sim}\right)$. Let $\frac{\{a_k\}_{k\in D}}{\sim} \in \widehat{I_L^{\star}}\left(\frac{\{x_i\}_{i\in D}}{\sim}\right)$ and $\frac{\{b_l\}_{l\in D}}{\sim} \in \widehat{I_L^{\star}}\left(\frac{\{y_j\}_{j\in D}}{\sim}\right)$. Then $\left(\frac{\{x_i\}_{i\in D}}{\sim}, \frac{\{a_k\}_{k\in D}}{\sim}\right) \in \widehat{I_L^{\star}}$ and $\left(\frac{\{y_j\}_{j\in D}}{\sim}, \frac{\{b_l\}_{l\in D}}{\sim}\right) \in \widehat{I_L^{\star}}$. Hence there exist $i_0, j_0, k_0$ and $l_0 \in D$ such that $(x_i, a_k) \in I_L^{\star}$ and $(y_j, b_l) \in I_L^{\star}$ for any $i \geq i_0, j \geq j_0, k \geq k_0$ and $l \geq l_0$. Thus $x_i \equiv^I a_k$ and $y_j \equiv^I b_l$ and so $x_i * y_j \equiv^I a_k * b_l$ for any $i \geq i_0, j \geq j_0, k \geq k_0$ and $l \geq l_0$. Therefore $(x_i * y_j, a_k * b_l) \in I_L^{\star}$ for any $i \geq i_0, j \geq j_0, k \geq k_0$ and $l \geq l_0$ and so $\left(\frac{\{x_i*y_j\}_{i,j\in D}}{\sim}, \frac{\{a_k*b_l\}_{k,l\in D}}{\sim}\right) \in \widehat{I_L^{\star}}$. Hence $\frac{\{a_k*b_l\}_{k,l\in D}}{\sim} \in \widehat{I_L^{\star}}\left(\frac{\{x_i*y_j\}_{i,j\in D}}{\sim}\right)$. Thus $\widehat{I_L^{\star}}\left(\frac{\{x_i\}_{i\in D}}{\sim}\right) * \widehat{I_L^{\star}}\left(\frac{\{y_j\}_{j\in D}}{\sim}\right) \subseteq \widehat{I_L^{\star}}\left(\frac{\{x_i*y_j\}_{i,j\in D}}{\sim}\right)$. □

**Definition 5.** [11] The uniform space $(A, Q)$ is *complete* if each cauchy net in $A$ is convergent.

**Definition 6.** [11] Let $(A, Q)$ be a uniform space. a uniform space $(\hat{A}, \hat{Q})$ is said to be a *completion* of $(A, Q)$ if

(i) $(\hat{A}, \hat{Q})$ is a complete uniform space.
(ii) $(A, Q)$ with its topology induced by its uniform structure is homeomorphic to a dense subspace of $(\hat{A}, \hat{Q})$.

**Theorem 7.** *The uniform space $(\mathcal{C}, \mathcal{V})$ is a completion of $(X, \mathcal{U}^{\star})$.*

**Proof.** Let $i : X \to \mathcal{C}$ be defined by $i(x) = \frac{\{x\}_{i\in D}}{\sim}$. Clearly, $i$ is one to one. We show that $i(X)$ is dense in $\mathcal{C}$. Let $\hat{U}\left(\frac{\{x_i\}_{i\in D}}{\sim}\right) \in T(\mathcal{V})$. Then

$$\begin{aligned}
\hat{U}\left(\frac{\{x_i\}_{i\in D}}{\sim}\right) \cap i(X) &= \left\{i(x) : \left(\frac{\{x_i\}_{i\in D}}{\sim}, i(x)\right) \in \hat{U}\right\}, \\
&= \{i(x) : \exists i_0 \in D\ \ \forall i \geq i_0\ s.t.\ \ (x_i, x) \in U\}, \\
&= \{i(x) : \exists i_0 \in D\ \ \forall i \geq i_0\ s.t.\ \ x \in U(x_i)\}, \\
&= \left\{i(x) : x \in \bigcup_{i\in D} \bigcap_{i_0 \leq i} U(x_i)\right\}, \\
&= i(V)
\end{aligned}$$

where $V = \bigcup_{i\in D} \bigcap_{i_0\leq i} U(x_i)$. Hence $\hat{U}\left(\frac{\{x_i\}_{i\in D}}{\sim}\right) \cap i(X) \neq \emptyset$ and so $i(X)$ is dense in $\mathcal{C}$. It is easy to see that $i : X \to i(X)$ is a homeomorphism. Now we show that the uniform space $(\mathcal{C}, \mathcal{V})$ is complete. Let $\left\{\frac{\{x_i^{\alpha}\}_{i\in D}}{\sim}\right\}_{\alpha\in D}$ be a cauchy net in $\mathcal{C}$. We have to show that it is convergent. Let $U \in \mathcal{U}^{\star}$. Since $\left\{\frac{\{x_i^{\alpha}\}_{i\in D}}{\sim}\right\}_{\alpha\in D}$ is a cauchy net, there exists $\gamma \in D$ such that $\left(\frac{\{x_i^{\alpha}\}_{i\in D}}{\sim}, \frac{\{x_i^{\beta}\}_{i\in D}}{\sim}\right) \in \hat{U}$ for any $\alpha, \beta \geq \gamma$. Hence there exist $\alpha_0, \beta_0 \in D$ such that $\left(x_i^{\alpha}, x_i^{\beta}\right) \in U$ for any $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$. We define the net of $\{y_j\}_{j\in D}$ by $y_j = x_i^{\beta_0}$ for any $j \in D$. Clearly, $\left(\frac{\{x_i^{\alpha}\}_{i\in D}}{\sim}, \frac{\{y_j\}_{j\in D}}{\sim}\right) \in \hat{U}$ for any $\alpha \geq \alpha_0$. Therefore $\left\{\frac{\{x_i^{\alpha}\}_{i\in D}}{\sim}\right\}_{\alpha\in D}$ is converges to $\frac{\{y_j\}_{j\in D}}{\sim}$. □

## 4. Conclusion

The aim of this paper was to study the concept of completion of a quasi-uniformity on a BCC-algebra. This work can be the basis for further and deeper research of the properties of BCC-algebras.

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