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On Jordan $(\theta, \phi)^*$-biderivations in Rings with Involution

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

Let R be a ring with involution. In the present paper, we characterize biadditive mappings which satisfies some functional identities related to symmetric Jordan $(\theta, \phi)^*$-biderivation of prime rings with involution. In particular, we prove that on a 2-torsion free prime ring with involution, every symmetric Jordan triple $(\theta, \phi)^*$-biderivation is a symmetric Jordan $(\theta, \phi)^*$-biderivation.

Keywords: Prime $*$-ring; involution; symmetric Jordan $(\theta, \phi)^*$-biderivation; symmetric Jordan triple $(\theta, \phi)^*$-biderivation.

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1 Introduction

Throughout the discussion, unless otherwise mentioned, R will denote an associative ring having at least two elements. However, R may not have unity. For any $x, y \in R$, the symbol $[x, y]$ (resp. $(x \circ y)$) will denote the commutator $xy - yx$ (resp. the anti-commutator $xy + yx$). Recall that R is prime if $aRb = 0$ implies that $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. An additive mapping $x \longmapsto x^*$ satisfying $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$, is called an involution on R. A ring R equipped with an involution is called $*$-ring or ring with involution.

An additive mapping $d : R \to R$ is called a derivation (resp. Jordan derivation) if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$) holds for all $x, y \in R$. An additive mapping $d : R \to R$ is a called Jordan triple derivation if $d(xyx) = d(x)yx + xd(y)x + xyd(x)$ holds for all $x, y \in R$. Of course every derivation is a Jordan triple derivation but the converse is not true in general. A classical result due to Brešar [[1],Theorem 4.3] asserts that any Jordan triple derivation on 2-torsion free semiprime ring is a derivation. Let R be a $*$-ring. An additive mapping $d : R \to R$ is said to be a $*$-derivation (resp. Jordan $*$-derivation) if $d(xy) = d(x)y^* + xd(y)$ (resp. $d(x^2) = d(x)x^* + xd(x)$) for all $x, y \in R$. These mappings appear naturally in the theory of representability of quadratic forms by bilinear forms. For results concerning this theory we refer the reader to [2] [3], [4], [5] and [6], where further references can be found. An additive mapping $d : R \longrightarrow R$ is said to be a Jordan triple $*$-derivation of R if $d(xyx) = d(x)y^*x^* + xd(y)x^* + xyd(x)$ holds for all $x, y \in R$. One can easily prove that every Jordan $*$-derivation on a 2-torsion free semiprime ring is a Jordan triple $*$-derivation of R. However, the converse of this statement need not be true in general. In [7], Vukman showed that the converse holds if R is 6-torsion free semiprime $*$-ring. Further, Fošner and Iliševic [8] generalized above mentioned result for 2-torsion free semiprime ring. et θ and ϕ be endomorphisms of R. An additive mapping $d : R \longrightarrow R$ is said to be a (θ, ϕ)-derivation (resp. Jordan (θ, ϕ)-derivation) if $d(xy) = d(x)\theta(y) + \phi(x)d(y)$ (resp. $d(x^2) = d(x)\theta(x) + \phi(x)d(x)$) holds for all $x, y \in R$. An additive mapping $d : R \to R$ is called $(\theta, \phi)^*$-derivation (resp. Jordan $(\theta, \phi)^*$-derivation) if $d(xy) = d(x)\theta(y^*) + \phi(x)d(y)$ (resp. $d(x^2) = d(x)\theta(x^*) + \phi(x)d(x)$) for all $x, y \in R$, where R is a ring with involution. Following [9], an additive mapping $d : R \to R$ is called Jordan triple $(\theta, \phi)^*$-derivation if $d(xyx) = d(x)\theta(y^*x^*) + \phi(x)d(y)\theta(x^*) + \phi(xy)d(x)$ for all $x, y \in R$. Obviously, every $(\theta, \phi)^*$-derivation on $*$-ring is a Jordan triple $(\theta, \phi)^*$-derivation but the converse is in general not true. Recently, first author together with Fošner [9] proved that on a 6-torsion free semiprime $*$-ring R, every Jordan triple $(\theta, \phi)^*$-derivation is a Jordan $(\theta, \phi)^*$-derivation. Further in [10], the first author improved this result by removing 3-torsion free restriction. More related results has also been obtained in [11], [12], [13], [14], [15], [16] and [17] where further references can be found.

A biaddive map $B : R \times R \to R$ is said to be symmetric if $B(x, y) = B(y, x)$ for all $x, y \in R$. A symmetric biadditive map $B : R \times R \to R$ is called a symmetric biderivation if $B(xy, z) = B(x, z)y + xB(y, z)$ is fulfilled for all $x, y, z \in R$. The concept of a symmetric biderivation was introduced by Maksa in [18] (see also [19], where an example can be found). A symmetric biadditive map $B : R \times R \to R$ is said to be a symmetric Jordan biderivation if $B(x^2, z) = B(x, z)x + xB(x, z)$ holds for all $x, z \in R$. Following [20], a symmetric biadditive map $B : R \times R \to R$ is called a symmetric $*$-biderivation if $B(xy, z) = B(x, z)y^* + xB(y, z)$ holds for all $x, y, z \in R$, where R is a ring with involution. In [12], Ali and Dar introduced the concept of symmetric Jordan $*$-biderivation and symmetric Jordan triple $*$-biderivation as follows: A symmetric biadditive map $d : R \times R \to R$ is said to be a symmetric Jordan $*$-biderivation if $d(x^2, z) = d(x, z)x^* + xd(x, z)$ holds for all $x, z \in R$. A symmetric biadditive map $d : R \times R \to R$ is called a symmetric Jordan triple $*$-biderivation if $d(xyx, z) = d(x, z)y^*x^* + xd(y, z)x^* + xyd(x, z)$ holds for all $x, y, z \in R$. Motivated by the definition of Jordan $(\theta, \phi)^*$-derivation and Jordan triple $(\theta, \phi)^*$- derivation, we introduce the concept of symmetric Jordan $(\theta, \phi)^*$-biderivation and symmetric Jordan triple $(\theta, \phi)^*$-biderivation as follows: A symmetric biadditive map $d : R\times R \to R$ is said to be a symmetric Jordan

$(\theta,\phi)^*$-biderivation if $d(x^2,z) = d(x,z)\theta(x^*) + \phi(x)d(x,z)$ holds for all $x,z \in R$. A symmetric biadditive map $d : R\times R \to R$ is called a symmetric Jordan triple $(\theta,\phi)^*$-biderivation if $d(xyx,z) = d(x,z)\theta(y^*x^*) + \phi(x)d(y,z)\theta(x^*) + \phi(xy)d(x,z)$ holds for all $x,y,z \in R$. Note that a symmetric Jordan triple $(I_R,I_R)^*$-biderivation is just a symmetyric Jordan triple $*$-biderivation, where I_R is the identity map on R. Clearly, this notion includes the notion of Jordan triple $*$-biderivation when $\theta = \phi = I_R$, where I_R is the identity map on R[see Lemma 1.2(ii)]. It can be easily seen that any symmetric Jordan $(\theta,\phi)^*$-biderivation on a 2-torsion free ring with involution is a symmetric Jordan triple $(\theta,\phi)^*$-biderivation. But the converse need not be true in general.

In the present paper, our aim is to establish a set of conditions under which every symmetric Jordan triple $(\theta,\phi)^*$-biderivation on a ring with involution is a symmetric Jordan $(\theta,\phi)^*$-biderivation. More precisely, we prove that on a 2-torsion free prime ring with involution, every symmetric Jordan triple $(\theta,\phi)^*$-biderivation is a symmetric Jordan $(\theta,\phi)^*$-biderivation.

In order to prove our main result we need to prove the following key lemma:

Lemma 1.1. *Let R be a prime ring with involution and θ,ϕ be automorphisms of R. For $a \in R$, if $\theta(x)a\phi(x^*) = 0$ for all $x \in R$, then $a = 0$.*

Proof. We have

$$\theta(x)a\phi(x^*) = 0 \text{ for all } x \in R. \tag{1.1}$$

Replacing x by $x^* + y$ in (1.1), we get

$$\theta(y)a\phi(x) + \theta(x^*)a\phi(y^*) = 0 \text{ for all } x,y \in R. \tag{1.2}$$

This can be further written as

$$\theta(y)a\phi(x) = -\theta(x^*)a\phi(y^*) \text{ for all } x,y \in R. \tag{1.3}$$

Applications of (1.1) and (1.3) yields that

$$\begin{aligned} a\theta(x)a\theta(z)a\phi(x)a &= a(\theta(x)a\theta(z))a\phi(x)a \\ &= -a\theta(z^*)a\theta(x^*)a\phi(x)a \\ &= -a\theta(z^*)a(\theta(x^*)a\phi(x))a \\ &= 0 \text{ for all } x,z \in R \end{aligned}$$

This implies that

$$a\theta(x)aRa\phi(x)a = (0) \text{ for all } x \in R.$$

The primeness of R forces that either $a\theta(x)a = 0$ or $a\phi(x)a = 0$ for all $x \in R$. Since θ and ϕ are automorphisms of R, so we are force to conclude that $aRa = (0)$. Hence, $a = 0$. This proves the lemma. □

Lemma 1.2. *Let R be a 2-torsion free ring with involution and θ,ϕ be endomorphisms of R. If $d : R\times R \to R$ is a symmetric Jordan $(\theta,\phi)^*$-biderivation of R, then the following hold:*

(i) $d(xy+yx,z) = d(x,z)\theta(y^*) + d(y,z)\theta(x^*) + \phi(x)d(y,z) + \phi(y)d(x,z)$ *for all* $x,y,z \in R$;

(ii) $d(xyx,z) = d(x,z)\theta(y^*x^*) + \phi(x)d(y,z)\theta(x^*) + \phi(xy)d(x,z)$ *for all* $x,y,z \in R$;

(iii) $d(xyt+tyx,z) = d(x,z)\theta(y^*t^*) + \phi(x)d(y,z)\theta(t^*) + \phi(xy)d(t,z) + d(t,z)\theta(y^*x^*) + \phi(t)d(y,z)\theta(x^*) + \phi(ty)d(x,z)$ *for all* $t,x,y,z \in R$.

Proof. (i) We are given that $d : R \times R \to R$ is a symmetric Jordan $(\theta, \phi)^*$-biderivation of R i.e.,

$$d(x^2, z) = d(x, z)\theta(x^*) + \phi(x)d(x, z)$$

for all $x, z \in R$. Replacing x by $x + y$ in above expression, we obtain

$$\begin{aligned} d((x+y)^2, z) &= d(x, z)\theta(x^*) + d(x, z)\theta(y^*) + d(y, z)\theta(x^*) \\ &+ d(y, z)\theta(y^*) + \phi(x)d(x, z) + \phi(y)d(x, z) \\ &+ \phi(x)d(y, z) + \phi(y)d(y, z) \end{aligned} \tag{1.4}$$

for all $x, y, z \in R$. Also, we have

$$\begin{aligned} d((x+y)^2, z) &= d(xy + yx, z) + d(x, z)\theta(x^*) + \phi(x)d(x, z) \\ &+ d(y, z)\theta(y^*) + \phi(y)d(y, z) \end{aligned} \tag{1.5}$$

for all $x, y, z \in R$. On comparing last two relations we get the required result.

(ii) Replacing y by $xy + yx$ in (i), we get

$$\begin{aligned} & d(x(xy + yx) + (xy + yx)x, z) \\ = \ & d(xy + yx, z)\theta(x^*) + d(x, z)\theta(x^*y^* + y^*x^*) \\ + \ & \phi(x)d(xy + yx, z) + \phi(xy + yx)d(x, z) \\ = \ & d(xy, z)\theta(x^*) + d(yx, z)\theta(x^*) + d(x, z)\theta(x^*y^*) \\ + \ & d(x, z)\theta(y^*x^*) + \phi(x)d(xy, z) + \phi(x)d(yx, z) \\ + \ & \phi(xy)d(x, z) + \phi(yx)d(x, z) \\ = \ & d(x, z)\theta(y^*x^*) + d(x, z)\theta(x^*y^*) + d(x, z)\theta(y^*x^*) \\ + \ & d(y, z)\theta((x^*)^2) + \phi(x)d(y, z)\theta(x^*) + \phi(y)d(x, z)\theta(x^*) \\ + \ & \phi(x)d(x, z)\theta(y^*) + \phi(x)d(y, z)\theta(x^*) + \phi(x^2)d(y, z) \\ + \ & \phi(xy)d(x, z) + \phi(xy)d(x, z) + \phi(yx)d(x, z) \end{aligned} \tag{1.6}$$

for all $x, y, z \in R$. On the other hand, we have

$$\begin{aligned} & d(x(xy + yx) + (xy + yx)x, z) \\ = \ & d(x^2y + yx^2, z) + 2d(xyx, z) \\ = \ & d(x, z)\theta(x^*y^*) + \phi(x)d(x, z)\theta(y^*) + d(y, z)\theta((x^*)^2) \\ + \ & \phi(x^2)d(y, z) + \phi(y)d(x, z)\theta(x^*) + \phi(yx)d(x, z) \\ + \ & 2d(xyx, z) \end{aligned} \tag{1.7}$$

for all $x, y, z \in R$. Comparing (1.6) and (1.7), we obtain

$$2d(xyx, z) = 2\{d(x, z)\theta(y^*x^*) + \phi(x)d(y, z)\theta(x^*) + \phi(xy)d(x, z)\} \quad \text{for all } x, y, z \in R.$$

Since R is 2-torsion free ring, the last expression yields the required result.

(iii) Putting $x+t$ instead of x in (ii), we get

$$\begin{aligned} & d((x+t)y(x+t),z) \\ = \;& d(x+t,z)\theta(y^*)\theta(x^*+t^*)+\phi(x+t)d(y,z)\theta(x^*+t^*) \\ + \;& \phi(x+t)\phi(y)d(x+t,z) \\ = \;& d(x,z)\theta(y^*x^*)+d(x,z)\theta(y^*t^*)+d(t,z)\theta(y^*x^*)+d(t,z)\theta(y^*t^*) \\ + \;& \phi(x)d(y,z)\theta(x^*)+\phi(x)d(y,z)\theta(t^*)+\phi(t)d(y,z)\theta(x^*)+\phi(t)d(y,z)\theta(t^*) \\ + \;& \phi(xy)d(x,z)+\phi(xy)d(t,z)+\phi(ty)d(x,z)+\phi(ty)d(t,z) \end{aligned}$$

for all $t,x,y,z \in R$. On the other hand, we have

$$\begin{aligned} & d((x+t)y(x+t),z) \\ = \;& d(xyx,z)+d(tyt,z)+d(xyt+tyx,z) \\ = \;& d(x,z)\theta(y^*x^*)+\phi(x)d(y,z)\theta(x^*)+\phi(xy)d(x,z) \\ + \;& d(t,z)\theta(y^*t^*)+\phi(t)d(y,z)\theta(t^*)+\phi(ty)d(t,z)+d(xyt+tyx,z) \end{aligned}$$

for all $t,x,y,z \in R$. From the last two relations, we conclude the desired result. This completes the proof.

□

We are now have enough informations to prove our main theorem:

Theorem 1.3. *Let R be a prime ring with involution such that $char(R) \neq 2$ and θ, ϕ be automorphisms of R. Then any symmetric Jordan triple $(\theta,\phi)^*$-biderivation $d: R\times R \to R$ is a symmetric Jordan $(\theta,\phi)^*$-biderivation.*

Proof. Assume that $d: R\times R\to R$ is a symmetric Jordan triple $(\theta,\phi)^*$-biderivation of R i.e.,

$$d(xyx,z)=d(x,z)\theta(y^*x^*)+\phi(x)d(y,z)\theta(x^*)+\phi(xy)d(x,z) \tag{1.8}$$

for all $x,y,z\in R$. In view of Lemma 1.2 (iii), we have

$$\begin{aligned} d(xyt+tyx,z) \;=\;& d(x,z)\theta(y^*t^*)+\phi(x)d(y,z)\theta(t^*)+\phi(xy)d(t,z) \\ +\;& d(t,z)\theta(y^*x^*)+\phi(t)d(y,z)\theta(x^*)+\phi(ty)d(x,z) \end{aligned}$$

for all $t,x,y,z\in R$. Thus, we obtain

$$\begin{aligned} d((xy)^2,z) \;=\;& d(xyxy,z)=d(xy(xy)+(xy)yx-xy^2x,z) \\ =\;& d(xy(xy)+(xy)yx,z)-d(xy^2x,z) \\ =\;& d(x,z)\theta((y^*)^2)\theta(x^*)+\phi(x)d(y,z)\theta(y^*x^*)+\phi(xy)d(xy,z) \\ +\;& d(xy,z)\theta(y^*x^*)+\phi(xy)d(y,z)\theta(x^*)+\phi(xy^2)d(x,z) \\ -\;& d(x,z)\theta((y^*)^2)\theta(x^*)-\phi(x)d(y^2,z)\theta(x^*)-\phi(xy^2)d(x,z) \end{aligned}$$

for all $x,y,z\in R$. This implies that

$$\begin{aligned} 0 \;=\;& d((xy)^2,z)-d(xy,z)\theta(y^*x^*)-\phi(xy)d(xy,z) \\ +\;& \phi(x)(d(y^2,z)-d(y,z)\theta(y^*)-\phi(y)d(y,z))\theta(x^*) \end{aligned} \tag{1.9}$$

for all $x,y,z\in R$. Thus, the relation (1.9) can be rewritten in the following form

$$\Delta(xy)+\phi(x)\Delta(y)\theta(x^*)=0 \tag{1.10}$$

for all $x,y\in R$, where

$$\Delta(x)=d(x^2,z)-d(x,z)\theta(x^*)-\phi(x)d(x,z)$$

for all $x, z \in R$. Application of relation (1.10) yields that

$$\begin{aligned} 2\phi(ty)\Delta(x)\theta(y^*t^*) &= \phi(ty)\Delta(x)\theta(y^*t^*) + \phi(ty)\Delta(x)\theta(y^*t^*) \\ &= -\phi(t)\Delta(yx)\theta(t^*) - \Delta((ty)x) \\ &= -\phi(t)\Delta(yx)\theta(t^*) - \Delta(tyx) \\ &= \Delta(tyx) - \Delta(tyx) \\ &= 0 \end{aligned}$$

for all $x, y, t \in R$. Thus $2\phi(ty)\Delta(x)\theta(y^*t^*) = 0$ for all $x, y, t \in R$. Since $char(R) \neq 2$, the above relation yields that $\phi(ty)\Delta(x)\theta(y^*t^*) = 0$ for all $x, y, t \in R$. Hence, application of Lemma 1.1 twice yields that $\Delta(x) = 0$ for all $x \in R$. That is, $d(x^2, z) - d(x, z)\theta(x^*) - \phi(x)d(x, z) = 0$ for all $x, z \in R$. Hence, d is a symmetric Jordan $(\theta, \phi)^*$-biderivation on R. This completes the proof of the theorem. □

From the above theorem, we now deduce immediate the following corollary.

Corollary 1.4. *Let R be a prime ring with involution such that $char(R) \neq 2$. Then every symmetric Jordan triple *-biderivation $d : R \times R \to R$ is a symmetric Jordan *-biderivation.*

2 Conclusion

In conclusion, we characterize biadditive mappings which satisfies some functional identities related to symmetric Jordan $(\theta, \phi)^*$-biderivation of prime rings. In particular, we prove that on a 2-torsion free prime ring with involution, every symmetric Jordan triple $(\theta, \phi)^*$-biderivation is a symmetric Jordan $(\theta, \phi)^*$-biderivation.

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Competing Interests

Authors have declared that no competing interests exist.

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