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A Clas of A-Stable Runge-Kutta Collocation Methods for the Solution of First Order Ordinary Differential Equations

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

This paper presented a class of A-stable Runge-Kutta collocation methods with three free parameters for the solution of first order ordinary differential equations. Power series was considered as its basis function, adoption of interpolation and collocation of the approximate solution at some selected grid points to give system of equations was also considered. Gaussian Elimination method was used to solve for the unknown parameters and substituted into the approximate solution to give the continuous method. The three cases considered are the Guass, the Lobatto, and the Radau types. Analysis of the methods was made based on order, zero stability, consistence and convergence. The derived schemes were implemented in the Predictor-Corrector mode. Comparison with existing methods showed that the new developed Schemes compete favorably.

Keywords: Runge kutta; interpolation; collocation; approximate solution; grid point; continuous method.

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1 Introduction

It is remarkable to note that many physical phenomena in sciences, engineering, and medicine, to mention but a few, are modeled by equations involving derivatives, which are generally referred to as differential equations. A differential equation in which the unknown parameter is a function of one independent variable is called an ordinary differential equations, while that involving two or more independent variables is called a partial differential equation. The general form of ordinary differential equations is in the form

$$y^n = F(x, y, y', y'', ..., y^{n-1}) \tag{1}$$

In most cases, modeled problems do not have analytical solutions; hence numerical methods are often the only option to solve such problems. Many physical problems are modeled into first order ordinary differential equations, the few that are modeled into higher order ordinary differential equations, are solved by reducing them to a system of first order ordinary differential equations. Hence, the study of first order ordinary differential equation is important. This paper considered a numerical method of solving first order initial value problems of ordinary differential equations of the form;

$$y' = f(x, y), y(x_0) = y_0 \tag{2}$$

where f is continuous and satisfies Lipschitz's condition, x_0 is the initial point and y_0 is the solution at x_0. There are a lot of numerical methods available in the literature for solving the problem in equation (2), the Runge-Kutta and multistep methods are well-known schemes that have being used largely for this purpose as reported in [1-3]. Adoption of collocation and interpolation of power series approximate solution for the solution of initial value problems have been studied by $[4-14]$, most of these methods failed when the problem is stiff or stiff oscillatory. Linear multistep method has been reported to be efficient and easier to implement for the solution of ordinary differential equations $[15-16]$. $[15-19]$ studied development of hybrid methods, they reported that hybrid methods were difficult to develop, but give methods with good stability properties due to the reduction in the step length. This paper introduced a new continuous Runge-Kutta collocation method with three free parameters using power series as the approximate solution. The introduction of the free parameters made this work different from the existing methods. Our interest also include investigation into how best to fix the free parameters, the three cases considered are; Guass type, Radau type and Lobatto type.

Section two discussed the research methodology, section three showed the analysis of the methods developed while section four showed the results of the implementation of the methods on some selected problems.

2 Methodology

We considered power series approximate solution of the form;

$$y(x) = \sum_{j=0}^{s+r-1} a_j x^j \tag{3}$$

where r and s are the number of interpolation and collocation points respectively, and $a_j's$ are parameters to be determined, x is continuous and differentiable.

First derivative of (3) gives the following

$$y'(x) = \sum_{j=0}^{s+r-1} j a_j x^{j-1} \tag{4}$$

substituting (4) in (2) gives

$$f(x,y) = \sum_{j=0}^{s+r-1} j a_j x^{j-1} \tag{5}$$

Interpolating (3) at $x_{n+s}, s = 0$ and collocation (5) at points x_{n+r}, gives a system of linear equation

$$AX = B \tag{6}$$

2.1 Derivation of method one

2.1.1 The gauss type:

$u = \frac{5-\sqrt{15}}{10}, v = \frac{5+\sqrt{15}}{10}, w = \frac{1}{2}$
Interpolating (3) at $x_{n+s}, s = 0$ and collocating (5) at $x_{n+r}, r = 0, \frac{5-\sqrt{15}}{10}, \frac{1}{6}, \frac{1}{2}, \frac{2}{3}, \frac{5+\sqrt{15}}{10}, 1$, gives a system of linear equations in (6)
where

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{bmatrix}^T$$

$$B = \begin{bmatrix} y_n & f_n & f_{n+\frac{5-\sqrt{15}}{10}} & f_{n+\frac{1}{6}} & f_{n+\frac{1}{2}} & f_{n+\frac{2}{3}} & f_{n+\frac{5+\sqrt{15}}{10}} & f_{n+1} \end{bmatrix}^T$$

$$X = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_{n+\frac{1}{6}} & 3x^2_{n+\frac{1}{6}} & 4x^3_{n+\frac{1}{6}} & 5x^4_{n+\frac{1}{6}} & 6x^5_{n+\frac{1}{6}} & 7x^6_{n+\frac{1}{6}} \\ 0 & 1 & 2x_{n+\frac{5-\sqrt{15}}{10}} & 3x^2_{n+\frac{5-\sqrt{15}}{10}} & 4x^3_{n+\frac{5-\sqrt{15}}{10}} & 5x^4_{n+\frac{5-\sqrt{15}}{10}} & 6x^5_{n+\frac{5-\sqrt{15}}{10}} & 7x^6_{n+\frac{5-\sqrt{15}}{10}} \\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x^2_{n+\frac{1}{2}} & 4x^3_{n+\frac{1}{2}} & 5x^4_{n+\frac{1}{2}} & 6x^5_{n+\frac{1}{2}} & 7x^6_{n+\frac{1}{2}} \\ 0 & 1 & 2x_{n+\frac{2}{3}} & 3x^2_{n+\frac{2}{3}} & 4x^3_{n+\frac{2}{3}} & 5x^4_{n+\frac{2}{3}} & 6x^5_{n+\frac{2}{3}} & 7x^6_{n+\frac{2}{3}} \\ 0 & 1 & 2x_{n+\frac{5+\sqrt{15}}{10}} & 3x^2_{n+\frac{5+\sqrt{15}}{10}} & 4x^3_{n+\frac{5+\sqrt{15}}{10}} & 5x^4_{n+\frac{5+\sqrt{15}}{10}} & 6x^5_{n+\frac{5+\sqrt{15}}{10}} & 7x^6_{n+\frac{5+\sqrt{15}}{10}} \\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} & 4x^3_{n+1} & 5x^4_{n+1} & 6x^5_{n+1} & 7x^6_{n+1} \end{bmatrix}$$

Solving (6) for the unknown coefficients using Gaussian elimination method gives an S-stage Runge-Kutta method of the form

$$y(x) = \alpha_0 y_n + h(\beta_0 F_0 + \beta_1 F_1 + \beta_2 F_2 + \beta_3 F_3 + \beta_4 F_4 + \beta_5 F_5 + \beta_6 F_6) \tag{7}$$

$$F_i = f(x_n + c_i h, Y_i)$$

$$Y_i = h \sum_{j=0}^{s} a_{ij} F_j$$

where s is the internal stage, with the condition

$$c_i = \sum_{i=0}^{s} \beta_i$$

$$\alpha_0 = 1 \tag{8}$$

$$\beta_0 = \frac{1}{140}(3600t^7 - 14000t^6 + 21644t^5 - 16870t^4 + 6930t^3 - 1435t^2 + 140t) \tag{9}$$

$$\beta_1 = \frac{1}{1225}\left(194400t^7 - 718200t^6 + 1025136t^5 - 697410t^4 + 219240t^3 - 22680t^2\right) \tag{10}$$

$$\begin{aligned}\beta_2 = -\frac{1}{126(17\sqrt{15}-45)}&\Big(108000\sqrt{15}t^7 - 21000(9+17\sqrt{15})t^6 + 4200(126+109\sqrt{15})t^5\\ &- 2625(201+109\sqrt{15})t^4 + 3500(63+25\sqrt{15})t^3 - 10500(3+\sqrt{15})t^2\Big)\end{aligned} \tag{11}$$

$$\beta_3 = -\frac{1}{315}\Big(21600t^7 - 71400t^6 + 87024t^5 - 46830t^4 + 10360t^3 - 840t^2) \tag{12}$$

$$\beta_4 = \frac{1}{1540}\left(97200t^7 - 302400t^6 + 342468t^5 - 170100t^4 + 35910t^3 - 2835t^2\right) \tag{13}$$

$$\begin{aligned}\beta_5 = -\frac{1}{9702(17\sqrt{15}-45)}&\Big(108000(212\sqrt{15}-765)t^7 - 840(101575\sqrt{15}-14913)t^6\\ &+ 4200(29534\sqrt{15}-110097)t^5 - 2625(33359\sqrt{15}-125997)t^4 + 3500(8513\sqrt{15}\\ &- 32481)t^3 + 10500(1401-365\sqrt{15})t^2\Big)\end{aligned} \tag{14}$$

$$\beta_6 = \frac{1}{350}\left(3600t^7 - 9800t^6 + 9884t^5 - 4515t^4 + 910t^3 - 70t^2\right) \tag{15}$$

Evaluating equation (7) after substituting equations (8) - (15) at t = 1 gives the discrete scheme

$$\begin{aligned}y_{n+1} = y_n + h\Bigg[&\frac{9}{140}f_n + \left(\frac{400-405\sqrt{15}}{9702}\right)f_{n+\frac{(5-\sqrt{15})}{10}} + \frac{486}{1225}f_{n+\frac{1}{6}} + \frac{86}{315}f_{n+\frac{1}{2}}\\ &+ \frac{243}{1540}f_{n+\frac{2}{3}} + \left(\frac{400+405\sqrt{15}}{9702}\right)f_{n+\frac{(5+\sqrt{15})}{10}} + \frac{9}{350}f_{n+1}\Bigg]\end{aligned} \tag{16}$$

With its internal stages given as;

$$Y_1 = y_n \tag{17}$$

$$\begin{aligned}Y_2 = y_n + h\Bigg[&\frac{29363}{816480}f_n + \left(\frac{2992700+759735\sqrt{15}}{37721376}\right)f_{n+\frac{(5-\sqrt{15})}{10}} - \frac{3887}{132300}f_{n+\frac{1}{6}}\\ &+ \frac{1717}{306180}f_{n+\frac{1}{2}} - \frac{631}{166320}f_{n+\frac{2}{3}} + \left(\frac{2992700-759735\sqrt{15})}{37721376}\right)f_{n+\frac{(5+\sqrt{15})}{10}} - \frac{1649}{4082400}f_{n+1}\Bigg]\end{aligned} \tag{18}$$

$$\begin{aligned}Y_3 = y_n + h\Bigg[&\left(\frac{1825-141\sqrt{15}}{35000}\right)f_n + \left(\frac{24004+309\sqrt{15})}{194040}\right)f_{n+\frac{(5-\sqrt{15})}{10}}\\ &+ \left(\frac{56025-18954\sqrt{15}}{306250}\right)f_{n+\frac{1}{6}} + \left(\frac{10225-2514\sqrt{15}}{78750}\right)f_{n+\frac{1}{2}}\\ &+ \left(\frac{35100-9477\sqrt{15}}{385000}\right)f_{n+\frac{2}{3}} + \left(\frac{333\sqrt{15}-1264}{17640}\right)f_{n+\frac{(5+\sqrt{15})}{10}}\\ &+ \left(\frac{348\sqrt{15}-1425}{175000}\right)f_{n+1}\Bigg]\end{aligned} \tag{19}$$

$$\begin{aligned}Y_4 = y_n + \Bigg[&\frac{71}{1120}f_n + \left(\frac{-7300-2715\sqrt{15}}{155232}\right)f_{n+\frac{(5-\sqrt{15})}{10}} + \frac{1917}{4900}f_{n+\frac{1}{6}} + \frac{277}{1260}f_{n+\frac{1}{2}}\\ &+ \frac{33}{5600}f_{n+\frac{2}{3}} - \left(\frac{2715\sqrt{15}-7300}{155232}\right)f_{n+\frac{(5+\sqrt{15})}{10}}\frac{459}{6160}f_{n+1}\Bigg]\end{aligned} \tag{20}$$

$$
\begin{aligned}
Y_5 = y_n + \bigg[& \frac{1531}{25515} f_n + \frac{11968}{33075} f_{n+\frac{(5-\sqrt{15})}{10}} + \frac{(-44600 - 15480\sqrt{15}}{1178793} f_{n+\frac{1}{6}} + \frac{23872}{76545} f_{n+\frac{1}{2}} \\
& + \frac{131}{10395} f_{n+\frac{2}{3}} + \left(\frac{15480\sqrt{15} - 4460)}{1178793} \right) f_{n+\frac{(5+\sqrt{15})}{10}} - \frac{508}{127575} f_{n+1} \bigg]
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
Y_6 = y_n + \bigg[& \left(\frac{1825 + 141\sqrt{15}}{35000} \right) f_n + \left(\frac{56025 + 18954\sqrt{15}}{306250} \right) f_{n+\frac{(5-\sqrt{15})}{10}} \\
& + \left(\frac{-1264 - 333\sqrt{15}}{17640} \right) f_{n+\frac{1}{6}} + \left(\frac{10225 + 2514\sqrt{15}}{78750} \right) f_{n+\frac{1}{2}} + \left(\frac{35100 + 9477\sqrt{15}}{385000} \right) f_{n+\frac{2}{3}} \\
& + \left(\frac{24004 - 309\sqrt{15}}{194040} \right) f_{n+\frac{(5+\sqrt{15})}{10}} + \left(\frac{-1425 - 348\sqrt{15}}{175000} \right) f_{n+1} \bigg]
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
Y_7 = y_n + h\bigg[& \frac{9}{140} f_n + \left(\frac{400 - 405\sqrt{15}}{9702} \right) f_{n+\frac{(5-\sqrt{15})}{10}} + \frac{486}{1225} f_{n+\frac{1}{6}} + \frac{86}{315} f_{n+\frac{1}{2}} \\
& + \frac{243}{1540} f_{n+\frac{2}{3}} + \left(\frac{400 + 405\sqrt{15}}{9702} \right) f_{n+\frac{(5+\sqrt{15})}{10}} + \frac{9}{350} f_{n+1} \bigg]
\end{aligned} \tag{23}
$$

To implement (7); [20] proposed a prediction equation in the form

$$
y_i = y_n + \sum_{i=1}^{k} (c_i)^i \left| \frac{\partial^i}{\partial x^i} f(x, y) \right|_{(x_n, y_n)} \tag{24}
$$

Equations (16)-(23) is written compactly in the partition Butcher's table of the form

$$
\begin{array}{c|c} c & A \\ \hline & b^T \end{array} \tag{25}
$$

0	0	0	0	0	0	0	0
$\frac{1}{6}$	$\frac{29363}{816480}$	$-\frac{3887}{132300}$	$\frac{2992700+759735\sqrt{15}}{37721376}$	$\frac{1717}{306180}$	$-\frac{631}{166320}$	$\frac{2992700-759735\sqrt{15}}{37721376}$	$-\frac{1649}{4082400}$
$\frac{5-\sqrt{15}}{10}$	$\frac{1825-141\sqrt{15}}{35000}$	$\frac{56025-18954\sqrt{15}}{306250}$	$\frac{24004+309\sqrt{15}}{194040}$	$\frac{10225-2514\sqrt{15}}{78750}$	$\frac{35100-9477\sqrt{15}}{385000}$	$\frac{-1264+333\sqrt{15}}{17640}$	$\frac{-1425+348\sqrt{15}}{175000}$
$\frac{1}{2}$	$\frac{71}{1120}$	$\frac{1917}{4900}$	$\frac{-7300-2715\sqrt{15}}{155232}$	$\frac{277}{1260}$	$-\frac{459}{6160}$	$\frac{-7300+2715\sqrt{15}}{155232}$	$-\frac{33}{5600}$
$\frac{2}{3}$	$\frac{1531}{25515}$	$\frac{11968}{33075}$	$\frac{-44600-15480\sqrt{15}}{1178793}$	$\frac{23872}{76545}$	$\frac{131}{10395}$	$\frac{-44600+15480\sqrt{15}}{1178793}$	$-\frac{508}{127575}$
$\frac{5+\sqrt{15}}{10}$	$\frac{1825+141\sqrt{15}}{35000}$	$\frac{56025+18954\sqrt{15}}{306250}$	$\frac{-1264-333\sqrt{15}}{17640}$	$\frac{10225+2514\sqrt{15}}{78750}$	$\frac{35100+9477\sqrt{15}}{385000}$	$\frac{24004-309\sqrt{15}}{194040}$	$\frac{-1425-348\sqrt{15}}{175000}$
1	$\frac{9}{140}$	$\frac{400-405\sqrt{15}}{9702}$	$\frac{486}{1225}$	$\frac{86}{315}$	$\frac{243}{1540}$	$\frac{400+405\sqrt{15}}{9702}$	$\frac{9}{350}$
	$\frac{9}{140}$	$\frac{486}{1225}$	$\frac{400-405\sqrt{15}}{9702}$	$\frac{86}{315}$	$\frac{243}{1540}$	$\frac{400+405\sqrt{15}}{9702}$	$\frac{9}{350}$

2.2 Derivation of method two

2.2.1 The Lobatto type:

$u = \frac{5-\sqrt{5}}{10}, v = \frac{5+\sqrt{5}}{10}, w = \frac{1}{2}$
Interpolating (3) at $x_{n+s}, s = 0$ and collocating (5) at $x_{n+r}, r = 0, \frac{5-\sqrt{5}}{10}, \frac{1}{6}, \frac{1}{2}, \frac{2}{3}, \frac{5+\sqrt{5}}{10}, 1$, gives a system of linear equation in (6)
where

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{bmatrix}^T$$

$$B = \begin{bmatrix} y_n & f_n & f_{n+\frac{5-\sqrt{5}}{10}} & f_{n+\frac{1}{6}} & f_{n+\frac{1}{2}} & f_{n+\frac{2}{3}} & f_{n+\frac{5+\sqrt{5}}{10}} & f_{n+1} \end{bmatrix}^T$$

$$X = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_{n+\frac{1}{6}} & 3x^2_{n+\frac{1}{6}} & 4x^3_{n+\frac{1}{6}} & 5x^4_{n+\frac{1}{6}} & 6x^5_{n+\frac{1}{6}} & 7x^6_{n+\frac{1}{6}} \\ 0 & 1 & 2x_{n+\frac{5-\sqrt{5}}{10}} & 3x^2_{n+\frac{5-\sqrt{5}}{10}} & 4x^3_{n+\frac{5-\sqrt{5}}{10}} & 5x^4_{n+\frac{5-\sqrt{5}}{10}} & 6x^5_{n+\frac{5-\sqrt{5}}{10}} & 7x^6_{n+\frac{5-\sqrt{5}}{10}} \\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x^2_{n+\frac{1}{2}} & 4x^3_{n+\frac{1}{2}} & 5x^4_{n+\frac{1}{2}} & 6x^5_{n+\frac{1}{2}} & 7x^6_{n+\frac{1}{2}} \\ 0 & 1 & 2x_{n+\frac{2}{3}} & 3x^2_{n+\frac{2}{3}} & 4x^3_{n+\frac{2}{3}} & 5x^4_{n+\frac{2}{3}} & 6x^5_{n+\frac{2}{3}} & 7x^6_{n+\frac{2}{3}} \\ 0 & 1 & 2x_{n+\frac{5+\sqrt{5}}{10}} & 3x^2_{n+\frac{5+\sqrt{5}}{10}} & 4x^3_{n+\frac{5+\sqrt{5}}{10}} & 5x^4_{n+\frac{5+\sqrt{5}}{10}} & 6x^5_{n+\frac{5+\sqrt{5}}{10}} & 7x^6_{n+\frac{5+\sqrt{5}}{10}} \\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} & 4x^3_{n+1} & 5x^4_{n+1} & 6x^5_{n+1} & 7x^6_{n+1} \end{bmatrix}$$

Solving (6) for the unknown coefficients using Gaussian elimination method gives an S-stage Runge-Kutta method of the form

$$y(x) = \alpha_0 y_n + h(\gamma_0 F_0 + \gamma_1 F_1 + \gamma_2 F_2 + \gamma_3 F_3 + \gamma_4 F_4 + \gamma_5 F_5 + \gamma_6 F_6) \tag{26}$$

$$F_i = f(x_n + c_i h, Y_i)$$
$$Y_i = h\sum_{j=0}^{s} a_{ij} F_j$$

where s is the internal stage, with the condition

$$c_i = \sum_{i=0}^{s} \gamma_i$$

$$\alpha_0 = 1 \tag{27}$$

$$\gamma_0 = \frac{1}{420}(5400t^7 - 21000t^6 + 33222t^5 - 27510t^4 + 12740t^3 - 3255t^2 + 420t) \tag{28}$$

$$\gamma_1 = -\frac{216}{1925}\left(900t^7 - 3325t^6 + 4872t^5 - 3570t^4 + 1330t^3 - 210t^2\right) \tag{29}$$

$$\begin{aligned} \gamma_2 = -\frac{25}{9706(17\sqrt{5} - 35)}\Big(&108000(212\sqrt{5} - 765)t^7 - 840(104575\sqrt{5} - 14013)t^6 \\ &+ 4200(29534\sqrt{5} - 110097)t^5 - 2625(33359\sqrt{5} - 125997)t^4 + \\ &3500(85783\sqrt{5} - 32491)t^3 + 10590(1461 - 365\sqrt{5})t^2\Big) \end{aligned} \tag{30}$$

$$\gamma_3 = -\frac{1}{315}\Big(21600t^7 - 71400t^6 + 87024t^5 - 46830t^4 + 10360t^3 - 840t^2) \tag{31}$$

$$\gamma_4 = \frac{1}{1540}\left(97200t^7 - 302400t^6 + 342468t^5 - 170100t^4 + 35910t^3 - 2835t^2\right) \tag{32}$$

$$\gamma_5 = -\frac{1}{9702(17\sqrt{5}-45)}\Big(108000(212\sqrt{5}-765)t^7 - 840(101575\sqrt{5}-14913)t^6 + 4200(29534\sqrt{5}-110097)t^5 - 2625(33359\sqrt{5}-125997)t^4 + 3500(8513\sqrt{5}-32481)t^3 + 10500(1401-365\sqrt{5})t^2\Big) \tag{33}$$

$$\gamma_6 = \frac{1}{350}\left(3600t^7 - 9800t^6 + 9884t^5 - 4515t^4 + 910t^3 - 70t^2\right) \tag{34}$$

Evaluating equation (26) after substituting equations (27) - (34) at $t = 1$ gives the discrete scheme

$$y_{n+1} = y_n + h\Bigg[\frac{17}{420}f_n + \frac{648}{1925}f_{n+\frac{1}{6}} + \frac{(475-270\sqrt{5})}{924}f_{n+\frac{(5-\sqrt{5})}{10}} + \frac{24}{35}f_{n+\frac{1}{2}} - \frac{81}{70}f_{n+\frac{2}{3}} + \frac{(475+270\sqrt{5})}{924}f_{n+\frac{(5+\sqrt{5})}{10}} + \frac{139}{2100}f_{n+1}\Bigg] \tag{35}$$

With the internal stages given as

$$Y_1 = y_n \tag{36}$$

$$Y_2 = y_n + h\Bigg[\frac{41173}{816480}f_n + \frac{87343}{415800}f_{n+\frac{1}{6}} + \left(\frac{-137575-149760\sqrt{5}}{3592512}\right)f_{n+\frac{(5-\sqrt{5})}{10}} + + \frac{15593}{204120}f_{n+\frac{1}{2}} - \frac{5609}{60480}f_{n+\frac{2}{3}} + \left(\frac{149760\sqrt{5}-137575}{3592512}\right)f_{n+\frac{(5+\sqrt{5})}{10}} - \frac{1649}{4082400}f_{n+1}\Bigg] \tag{37}$$

$$Y_3 = y_n + h\Bigg[\left(\frac{5275-37\sqrt{5}}{105000}\right)f_n + \left(\frac{59400+1296\sqrt{5}}{240625}\right)f_{n+\frac{1}{6}} + \left(\frac{59-287\sqrt{5}}{9240}\right)f_{n+\frac{(5-\sqrt{5})}{10}} + \left(\frac{800-232\sqrt{5}}{4735}\right)f_{n+\frac{1}{2}} + \left(\frac{-675-324\sqrt{5}}{17500}\right)f_{n+\frac{2}{3}} + \left(\frac{491-23\sqrt{5}}{9274}\right)f_{n+\frac{(5+\sqrt{5})}{10}} + \left(\frac{31\sqrt{5}-475}{525000}\right)f_{n+1}\Bigg] \tag{38}$$

$$Y_4 = y_n + \Bigg[\frac{173}{3360}f_n + \frac{3537}{15400}f_{n+\frac{1}{6}} + \left(\frac{1175-60\sqrt{5}}{14784}\right)f_{n+\frac{(5-\sqrt{5})}{10}} + \frac{61}{280}f_{n+\frac{1}{2}} - \frac{351}{2240}f_{n+\frac{2}{3}} - \left(\frac{1175+60\sqrt{5}}{14784}\right)f_{n+\frac{(5+\sqrt{5})}{10}} - \frac{43}{33600}f_{n+1}\Bigg] \tag{39}$$

$$Y_5 = y_n + \Bigg[\frac{1301}{25515}f_n + \frac{12224}{51975}f_{n+\frac{1}{6}} + \left(\frac{3050+90\sqrt{5}}{56133}\right)f_{n+\frac{(5-\sqrt{5})}{10}} + \frac{7744}{25515}f_{n+\frac{1}{2}} - \frac{29}{945}f_{n+\frac{2}{3}} + \left(\frac{15480\sqrt{15}+4460)}{1178793}\right)f_{n+\frac{(5+\sqrt{5})}{10}} - \frac{128}{127575}f_{n+1}\Bigg] \tag{40}$$

$$Y_6 = y_n + \Bigg[\left(\frac{5275+37\sqrt{5}}{105000}\right)f_n + \left(\frac{59400-1296\sqrt{5}}{240625}\right)f_{n+\frac{1}{6}} + \left(\frac{491+23\sqrt{5}}{9240}\right)f_{n+\frac{(5-\sqrt{15})}{10}} + \left(\frac{800+232\sqrt{5}}{4375}\right)f_{n+\frac{1}{2}} + \left(\frac{324\sqrt{5}-675}{17500}\right)f_{n+\frac{2}{3}} + \left(\frac{59+287\sqrt{5}}{9240}\right)f_{n+\frac{(5+\sqrt{15})}{10}} + \left(\frac{-475-31\sqrt{5}}{25000}\right)f_{n+1}\Bigg] \tag{41}$$

$$Y_7 = y_n + h\left[\frac{17}{420}f_n + \frac{648}{1925}f_{n+\frac{1}{6}} + \frac{(475-270\sqrt{5})}{924}f_{n+\frac{(5-\sqrt{5})}{10}} + \frac{24}{35}f_{n+\frac{1}{2}} + \frac{139}{2100}f_{n+\frac{2}{3}} + \frac{(475+270\sqrt{5})}{924}f_{n+\frac{(5+\sqrt{5})}{10}} - \frac{81}{70}f_{n+1}\right] \tag{42}$$

To implement (26); according to [14] shown in equation (24), the derived scheme and its internal stages can be written compactly in a partitioned Butcher's table of the form

0	0	0	0	0	0	0	0
$\frac{1}{6}$	$\frac{41173}{816480}$	$\frac{87343}{415800}$	$\frac{-137575-149760\sqrt{5}}{3592512}$	$\frac{15593}{204120}$	$-\frac{5609}{60480}$	$\frac{-137575+149760\sqrt{5}}{3592512}$	$-\frac{7193}{8164800}$
$\frac{5-\sqrt{5}}{10}$	$\frac{5275-37\sqrt{5}}{105000}$	$\frac{59400+1296\sqrt{5}}{240625}$	$\frac{59-287\sqrt{5}}{9240}$	$\frac{800-232\sqrt{5}}{4375}$	$\frac{-675-324\sqrt{5}}{17500}$	$\frac{491-23\sqrt{5}}{9240}$	$\frac{-475+31\sqrt{5}}{525000}$
$\frac{1}{2}$	$\frac{173}{3360}$	$\frac{3537}{15400}$	$\frac{1175-60\sqrt{5}}{14784}$	$\frac{61}{280}$	$-\frac{351}{2240}$	$\frac{1175+60\sqrt{5}}{14784}$	$-\frac{43}{33600}$
$\frac{2}{3}$	$\frac{1301}{25515}$	$\frac{12224}{51975}$	$\frac{3050+90\sqrt{5}}{56133}$	$\frac{7744}{25515}$	$-\frac{29}{945}$	$\frac{3050-90\sqrt{5}}{56133}$	$-\frac{128}{127575}$
$\frac{5+\sqrt{5}}{10}$	$\frac{5275+37\sqrt{5}}{105000}$	$\frac{59400-1296\sqrt{5}}{240625}$	$\frac{491+23\sqrt{5}}{9240}$	$\frac{800+232\sqrt{5}}{4375}$	$\frac{-675+324\sqrt{5}}{17500}$	$\frac{59+287\sqrt{5}}{9240}$	$\frac{-475-31\sqrt{5}}{525000}$
1	$\frac{17}{420}$	$\frac{648}{1925}$	$\frac{475-270\sqrt{5}}{924}$	$\frac{24}{35}$	$-\frac{81}{70}$	$\frac{475+270\sqrt{5}}{924}$	$\frac{139}{2100}$
	$\frac{17}{420}$	$\frac{648}{1925}$	$\frac{475-270\sqrt{5}}{924}$	$\frac{24}{35}$	$-\frac{81}{70}$	$\frac{475+270\sqrt{5}}{924}$	$\frac{139}{2100}$

2.3 Derivation of method three

2.3.1 The Radau type:

$u = \frac{6-\sqrt{6}}{10}, v = \frac{6+\sqrt{6}}{10}, w = \frac{1}{2}$

Interpolating (3) at $x_{n+s}, s = 0$ and collocating (5) at $x_{n+r}, r = 0, \frac{6-\sqrt{6}}{10}, \frac{1}{6}, \frac{1}{2}, \frac{2}{3}, \frac{6+\sqrt{6}}{10}, 1$, gives a system of linear equation in (6)

where

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \end{bmatrix}^T$$

$$B = \begin{bmatrix} y_n & f_n & f_{n+\frac{5-\sqrt{5}}{10}} & f_{n+\frac{1}{6}} & f_{n+\frac{1}{2}} & f_{n+\frac{2}{3}} & f_{n+\frac{5+\sqrt{5}}{10}} & f_{n+1} \end{bmatrix}^T$$

$$X = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 \\ 0 & 1 & 2x_{n+\frac{1}{6}} & 3x_{n+\frac{1}{6}}^2 & 4x_{n+\frac{1}{6}}^3 & 5x_{n+\frac{1}{6}}^4 & 6x_{n+\frac{1}{6}}^5 & 7x_{n+\frac{1}{6}}^6 \\ 0 & 1 & 2x_{n+\frac{6-\sqrt{6}}{10}} & 3x_{n+\frac{6-\sqrt{6}}{10}}^2 & 4x_{n+\frac{6-\sqrt{6}}{10}}^3 & 5x_{n+\frac{6-\sqrt{6}}{10}}^4 & 6x_{n+\frac{6-\sqrt{6}}{10}}^5 & 7x_{n+\frac{6-\sqrt{6}}{10}}^6 \\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x_{n+\frac{1}{2}}^2 & 4x_{n+\frac{1}{2}}^3 & 5x_{n+\frac{1}{2}}^4 & 6x_{n+\frac{1}{2}}^5 & 7x_{n+\frac{1}{2}}^6 \\ 0 & 1 & 2x_{n+\frac{2}{3}} & 3x_{n+\frac{2}{3}}^2 & 4x_{n+\frac{2}{3}}^3 & 5x_{n+\frac{2}{3}}^4 & 6x_{n+\frac{2}{3}}^5 & 7x_{n+\frac{2}{3}}^6 \\ 0 & 1 & 2x_{n+\frac{6+\sqrt{6}}{10}} & 3x_{n+\frac{6+\sqrt{6}}{10}}^2 & 4x_{n+\frac{6+\sqrt{6}}{10}}^3 & 5x_{n+\frac{6+\sqrt{6}}{10}}^4 & 6x_{n+\frac{6+\sqrt{6}}{10}}^5 & 7x_{n+\frac{6+\sqrt{6}}{10}}^6 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & 7x_{n+1}^6 \end{bmatrix}$$

Solving (6) for the unknown coefficients using Gaussian elimination method gives an S-stage Runge-Kutta method of the form

$$y(x) = \alpha_0 y_n + h(\phi_0 F_0 + \phi_1 F_1 + \phi_2 F_2 + \phi_3 F_3 + \phi_4 F_4 + \phi_5 F_5 + \phi_6 F_6) \tag{43}$$

$$F_i = f(x_n + c_i h, Y_i)$$

$$Y_i = h \sum_{j=0}^{s} a_{ij} F_j$$

where s is the internal stage, with the condition

$$c_i = \sum_{i=0}^{s} \phi_i$$

$$\alpha_0 = 1 \tag{44}$$

$$\phi_0 = \frac{1}{1260}(10800t^7 - 13400t^6 + 223644t^5 - 16450t^4 + 6930t^3 - 1435t^2 + 140t) \tag{45}$$

$$\phi_1 = \frac{1}{12345}\left(174400t^7 - 818200t^6 + 340236t^5 - 697410t^4 + 234240t^3 - 22645t^2\right) \tag{46}$$

$$\begin{aligned} \phi_2 = &-\frac{1}{123(17\sqrt{6}-45)}\Big(109000\sqrt{6}t^7 - 31000(9+17\sqrt{6})t^6 + 4200(126+109\sqrt{6})t^5 \\ &- 2625(201+109\sqrt{6})t^4 + 3500(63+25\sqrt{6})t^3 - 20500(3+\sqrt{6})t^2\Big) \end{aligned} \tag{47}$$

$$\phi_3 = -\frac{1}{315}\Big(21600t^7 - 71400t^6 + 87024t^5 - 462330t^4 + 10360t^3 - 840t^2) \tag{48}$$

$$\phi_4 = \frac{1}{1540}\left(97200t^7 - 302400t^6 + 34468t^5 - 170100t^4 + 3520t^3 - 2835t^2\right) \tag{49}$$

$$\begin{aligned} \phi_5 = &-\frac{1}{9702(17\sqrt{6}-45)}\Big(10800(212\sqrt{6}-765)t^7 - 840(101575\sqrt{6} - 14913)t^6 \\ &+ 4200(29534\sqrt{6} - 110097)t^5 - 2625(33359\sqrt{6} - 125997)t^4 + 3500(8513\sqrt{6} - 32481)t^3 \\ &+ 10500(1401 - 365\sqrt{6})t^2\Big) \end{aligned} \tag{50}$$

$$\phi_6 = \frac{1}{450}\left(5600t^7 - 55800t^6 + 9544t^5 - 4505t^4 + 910t^3 - 70t^2\right) \tag{51}$$

Evaluating equation (43) after successful substitution of equations (44) - (51) at $t = 1$ gives the discrete scheme

$$y_{n+1} = y_n + h\Big[\frac{31}{630}f_n + \frac{1026}{4025}f_{n+\frac{1}{6}} + \frac{(20816-544\sqrt{6})}{144900}f_{n+\frac{(6-\sqrt{6})}{10}} + \frac{2}{7}f_{n+\frac{1}{2}} + \frac{27}{350}f_{n+\frac{2}{3}} + \frac{(20816+544\sqrt{6})}{144900}f_{n+\frac{(6+\sqrt{6})}{10}} + \frac{8}{175}f_{n+1}\Big] \quad (52)$$

With the internal stages given as

$$Y_1 = y_n \quad (53)$$

$$Y_2 = y_n + h\Big[\frac{260431}{260431}f_n + \frac{18827}{108675}f_{n+\frac{1}{6}} + \Big(\frac{-34362392-18196337\sqrt{6}}{563371200}\Big)f_{n+\frac{(6-\sqrt{6})}{10}} + \frac{1291}{10206}f_{n+\frac{1}{2}} - \frac{18547}{302400}f_{n+\frac{2}{3}}\Big(\frac{-34362392+18196337\sqrt{6}}{563371200}\Big)f_{n+\frac{(6+\sqrt{6})}{10}} - \frac{2969}{1020600}f_{n+1}\Big] \quad (54)$$

$$Y_3 = y_n + h\Big[\Big(\frac{53127-172\sqrt{6}}{1050000}\Big)f_n + \Big(\frac{6184107+33048\sqrt{6}}{25156250}\Big)f_{n+\frac{1}{6}} + \Big(\frac{10272-2203\sqrt{6}}{193200}\Big)f_{n+\frac{(6-\sqrt{6})}{10}} + \Big(\frac{27063-6168\sqrt{6}}{218750}\Big)f_{n+\frac{1}{2}} + \Big(\frac{643707-380052\sqrt{6}}{8750000}\Big)f_{n+\frac{2}{3}} + \Big(\frac{33991392-11290337\sqrt{6}}{603750000}\Big)f_{n+\frac{(6+\sqrt{6})}{10}} + \Big(\frac{639\sqrt{6}-3474}{1093750}\Big)f_{n+1}\Big] \quad (55)$$

$$Y_4 = y_n + \Big[\frac{1021}{20160}f_n + \frac{1970}{8050}f_{n+\frac{1}{6}} + \Big(\frac{127048+39253\sqrt{6}}{2318400}\Big)f_{n+\frac{(6-\sqrt{6})}{10}} + \frac{1}{7}f_{n+\frac{1}{2}} - \frac{513}{11200}f_{n+\frac{2}{3}} - \Big(\frac{127048-39253\sqrt{6}}{2318400}\Big)f_{n+\frac{(6+\sqrt{6})}{10}} - \frac{3}{1400}f_{n+1}\Big] \quad (56)$$

$$Y_5 = y_n + \Big[\frac{3841}{76545}f_n + \frac{27008}{108675}f_{n+\frac{1}{6}} + \Big(\frac{360328+123208\sqrt{6}}{8802675}\Big)f_{n+\frac{(6-\sqrt{6})}{10}} + \frac{1280}{5103}f_{n+\frac{1}{2}} + \frac{173}{4725}f_{n+\frac{2}{3}} + \Big(\frac{360328-123208\sqrt{6}}{8802675}\Big)f_{n+\frac{(6+\sqrt{6})}{10}} - \frac{172}{12757}f_{n+1}\Big] \quad (57)$$

$$Y_6 = y_n + \Big[\Big(\frac{53127+172\sqrt{6}}{1050000}\Big)f_n + \Big(\frac{6184107-33048\sqrt{6}}{25156250}\Big)f_{n+\frac{1}{6}} + \Big(\frac{33991392+11290337\sqrt{6}}{603750000}\Big)f_{n+\frac{(6-\sqrt{6})}{10}} + \Big(\frac{27063+6168\sqrt{6}}{218750}\Big)f_{n+\frac{1}{2}} + \Big(\frac{643707+380052\sqrt{6}}{8750000}\Big)f_{n+\frac{2}{3}} + \Big(\frac{10272+2203\sqrt{6}}{193200}\Big)f_{n+\frac{(6+\sqrt{6})}{10}} + \Big(\frac{-3474-639\sqrt{6}}{1093750}\Big)f_{n+1}\Big] \quad (58)$$

$$Y_7 = y_n + h\Big[\frac{31}{630}f_n + \frac{1026}{4025}f_{n+\frac{1}{6}} + \frac{(2086-544\sqrt{6})}{144900}f_{n+\frac{(6-\sqrt{6})}{10}} + \frac{2}{7}f_{n+\frac{1}{2}} + \frac{27}{350}f_{n+\frac{2}{3}} + \frac{(2086+544\sqrt{6})}{144900}f_{n+\frac{(6+\sqrt{6})}{10}} + \frac{8}{175}f_{n+1}\Big] \quad (59)$$

To implement (43); following [14] as shown in equation (24), the derived scheme and its internal stages can be written compactly in a partitioned Butcher's table of the form;

0	0	0	0	0	0	0	0
$\frac{1}{6}$	$\frac{260431}{4898880}$	$\frac{18827}{108675}$	$\frac{-34362392-18196337\sqrt{6}}{563371200}$	$+\frac{1291}{10206}$	$-\frac{18547}{302400}$	$\frac{-34362392+18196337\sqrt{6}}{563371200}$	$-\frac{2969}{1020600}$
$\frac{6-\sqrt{6}}{10}$	$\frac{53127-172\sqrt{6}}{1050000}$	$\frac{6184107+33048\sqrt{6}}{25156250}$	$\frac{10272-2203\sqrt{6}}{193200}$	$\frac{27063-6168\sqrt{6}}{218750}$	$\frac{643707-380052\sqrt{6}}{8750000}$	$\frac{33991392-11290337\sqrt{6}}{603750000}$	$\frac{-3474+639\sqrt{6}}{1093750}$
$\frac{1}{2}$	$\frac{1021}{20160}$	$\frac{1970}{8050}$	$\frac{127048+39253\sqrt{6}}{2318400}$	$\frac{1}{7}$	$-\frac{513}{11200}$	$\frac{127048-39253\sqrt{6}}{2318400}$	$-\frac{3}{1400}$
$\frac{2}{3}$	$\frac{3841}{76545}$	$\frac{27008}{108675}$	$\frac{360328+123208\sqrt{6}}{8802675}$	$\frac{1280}{5103}$	$\frac{173}{4725}$	$\frac{360328-123208\sqrt{6}}{8802675}$	$-\frac{172}{127575}$
$\frac{6+\sqrt{6}}{10}$	$\frac{53127+172\sqrt{6}}{1050000}$	$\frac{6184107-33048\sqrt{6}}{25156250}$	$\frac{33991392+11290337\sqrt{6}}{603750000}$	$\frac{27063+6168\sqrt{6}}{218750}$	$\frac{643707+380052\sqrt{6}}{8750000}$	$\frac{10272+2203\sqrt{6}}{193200}$	$\frac{-3474-639\sqrt{6}}{1093750}$
1	$\frac{31}{630}$	$\frac{1026}{4025}$	$\frac{20816-5449\sqrt{6}}{144900}$	$\frac{2}{7}$	$\frac{27}{350}$	$\frac{20816+5449\sqrt{6}}{144900}$	$\frac{8}{175}$
	$\frac{31}{630}$	$\frac{1026}{4025}$	$\frac{20816-5449\sqrt{6}}{144900}$	$\frac{2}{7}$	$\frac{27}{350}$	$\frac{20816+5449\sqrt{6}}{144900}$	$\frac{8}{175}$

2.4 Implementation

To implement the methods; [14] proposed a prediction equation as shown in equation (24) which was employed in Predictor-Corrector mode to obtain the same order of accuracy. The following symmetric explicit predictor scheme of the same order with the corrector scheme are obtained using the same procedure for y_{n+1} of the the three schemes respectively;

$$y_{n+1} = \frac{59}{11}y_n - \frac{48}{11}y_{n+\frac{1}{2}} + h\left[\frac{15}{44}f_n + \frac{162}{77}f_{n+\frac{1}{6}} + \frac{(-1250-900\sqrt{15})}{7623}f_{n+\frac{(5-\sqrt{15})}{10}} + \frac{122}{99}f_{n+\frac{1}{2}} - \frac{81}{484}f_{n+\frac{2}{3}} + \frac{900\sqrt{15}-1250}{7623}f_{n+\frac{(5+\sqrt{15})}{10}}\right] \quad (60)$$

$$y_{n+1} = \frac{2267}{43}y_n - \frac{2224}{43}y_{n+\frac{1}{2}} + h\left[\frac{465}{172}f_n + \frac{5778}{473}f_{n+\frac{1}{6}} + \frac{(4375-475\sqrt{5})}{946}f_{n+\frac{(5-\sqrt{5})}{10}} + \frac{514}{43}f_{n+\frac{1}{2}} - \frac{1593}{172}f_{n+\frac{2}{3}} + \frac{(4375+475\sqrt{5})}{946}f_{n+\frac{(5+\sqrt{5})}{10}}\right] \quad (61)$$

$$y_{n+1} = \frac{67}{3}y_n - \frac{64}{3}y_{n+\frac{1}{2}} + h\left[\frac{61}{54}f_n + \frac{126}{23}f_{n+\frac{1}{6}} + \frac{(16304+4019\sqrt{6})}{12420}f_{n+\frac{(6-\sqrt{6})}{10}} + \frac{10}{3}f_{n+\frac{1}{2}} - \frac{9}{3}f_{n+\frac{2}{3}} + \frac{(16304+4019\sqrt{6})}{12420}f_{n+\frac{(6-\sqrt{6})}{10}}\right] \quad (62)$$

3 Analysis of the basic properties of the method

3.1 Order of the method

Let the linear operator $L\{y(x):h\}$ associated with the linear multistep method Equation (15) be defined as:

$$L\{y(x):h\} = y(x+h) - \alpha_0 y_n - h\sum_{j=0}^{6}\beta_j(x)f_{n+j} \quad (63)$$

Expanding in Taylor series and comparing the coefficients of
$L\{y(x):h\} = c_0y(x) + c_1hy'(x) + c_2h^2y''(x) + ... + c_ph^py^p(x) + c_{p+1}y^{p+1}y^{p+1}(x) + ...$

Defination 1:
According to Lambert (1973); The linear multistep method is said to be of order p if $c_0 = c_1 = c_2 = c_3 = c_p = 0$ and $c_{p+1} \neq 0$ is called the error constant which implies the local truncation error is given by

$$t_{n+k} = c_{p+1}h^{p+1}y^{p+1}(x_n) + Oh^{p+1} \quad (64)$$

From (16)
$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = 0; c_8 \neq 0$, therefore the Method is of order 7 with error constant $\frac{1}{84672000}$.
From(41)
$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = 0; c_8 \neq 0$, therefore the Method is of order 7 with error constant $-\frac{1}{63504000}$
From(62)
$c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = 0; c_8 \neq 0$, therefore the Method is of order 7 with error constant $\frac{1}{762048000}$

3.2 Consistency

Defination: A linear multistep method is said to be consistent if it satisfies the following conditions according to Lambert (1973):

1. It has order p ≥ 1
2. $\sum_{j=0}^{k} \alpha_j = 0$
3. $\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j$

Since the three conditions are satisfied, therefore, our method is consistent.

3.3 Zero Stability

A linear multistep method is said to be zero stable if the zeros of the first characteristic polynomial $\rho(r)$ satisfy$|r| \leq 1$ and the roots $|r| \leq 1$ have multiplicity not exceeding the order of the differential equation see Lambert (1973), hence our method is zero-stable

3.4 Covergency

The necessary and sufficient condition for a linear multistep method to be convergent is that it must be consistent and zero stable, hence our method is convergent according to Lambert (1973)

3.5 Stability Region Plot

A method is said to be absolutely stable if for a given value of h, all the roots z_s of the characteristics polynomial $\prod(z, \bar{h}) = \rho(z) + \bar{h}\sigma(z) = 0$; satisfying $|z_s| \leq s, s = 1, 2, .., n$ where $\bar{h} = \lambda h, \lambda = \frac{df}{dy}$, substituting the test equation $y' = \lambda y$ and solving for $h = \lambda h$ and writing $r = e^{i\theta}$, gives the stability region as shown below;

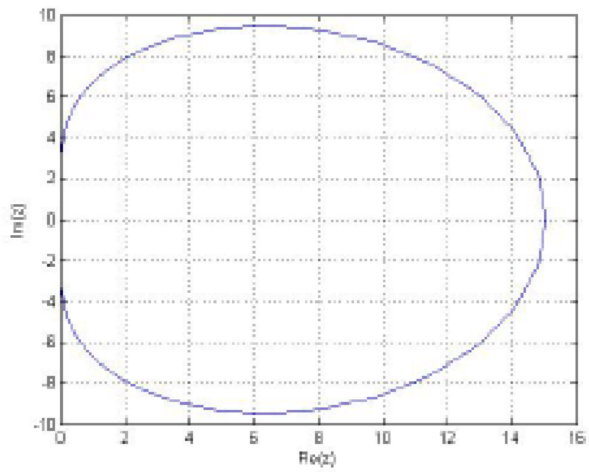

Fig. 1. Stability region for Gauss

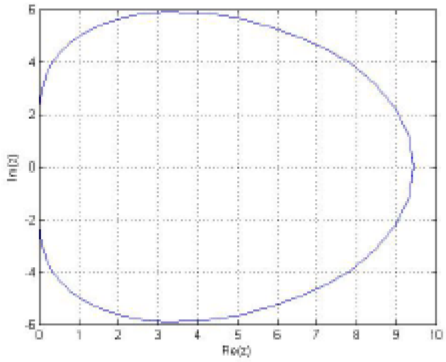

Fig. 2. Stability region for Lobatto

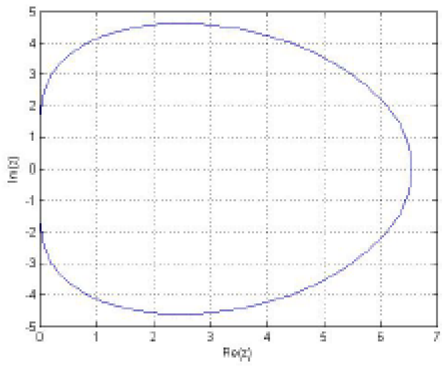

Fig. 3. Stability region for Radau

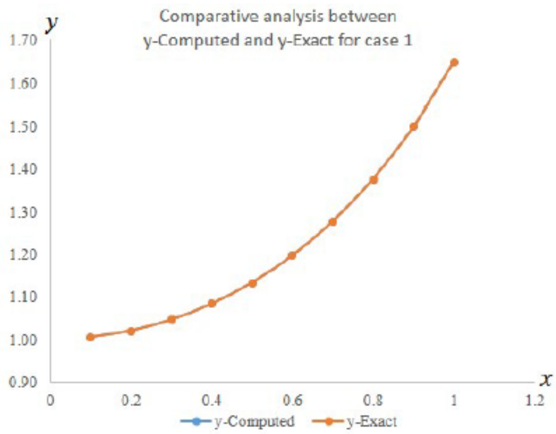

Fig. 4. Comparative analysis between y-Computed and y-Exact for case 1

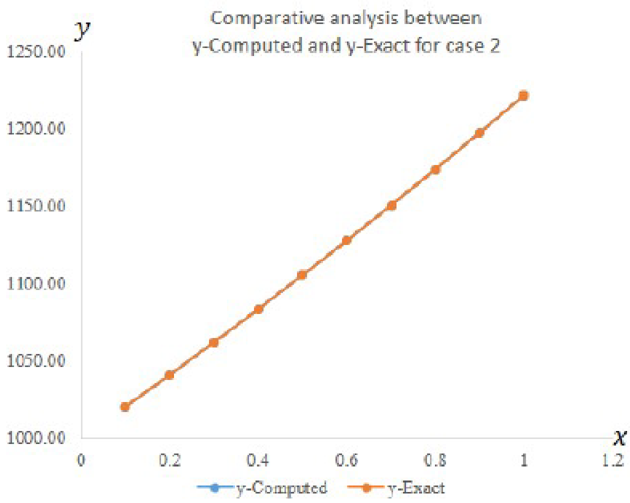

Fig. 5. Comparative analysis between y-Computed and y-Exact for case 2

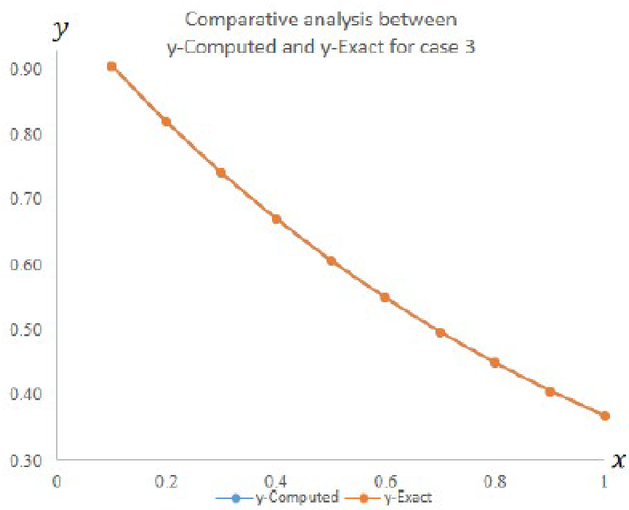

Fig. 6. Comparative analysis between y-Computed and y-Exact for case 3

A numerical method is said to be A-stable if the whole of the left-half plane $\{Z : Re(z) \leq 0\}$ is contained in the region $\{Z : Re(z) \leq 1\}$, where $R(z)$ is called the stability polynomial of the method see Lambert (1973).

Theorem 1: According to Butcher (2003), a Runge-Kutta method is A-stable only if allpoles of the stability function $R(z)$ lie in the right half-plane and no up arrow of the order web intersects with or is tangential to the imaginary axis.

Proof: See Butcher (2003)

Therefore figures 1,2 and 3 showed that the schemes are A-stable.

4 Numerical Example

The following are used in the tables
$Error = |y(x) - y(x_n)|$ where $y(x)$ is the exact result and $y(x_n)$ is the computed result.
Error 1 = Error in Gauss Scheme
Error 2 = Error in Lobatto Scheme
Error 3 = Error in Radau Scheme

4.1 Problem 1

Consider the ordinary differential equation:

$y' = xy,\ y(0) = 1,\ 0 \leq x \leq 1,\ h = 0.1,$

Exact solution $y(x) = e^{\frac{1}{2}x^2}$,

This problem was solved by James A. A *et al.*(2013)

4.2 Problem 2

We Consider the growth model described by the differential equation of the form

$\frac{dN}{dt} = \alpha N,\ \ N(0) = 1000,\ \ t \ \epsilon \ [0, 1]$

The above growth equation represents the rate of growth of bacteria in a colony. We shall assume that the model grows continuously without restriction. One may ask; how many bacteria are in a colony after some minutes if an individual produces an offspring at an average growth rate of 0.2? We also assume that $N(t)$ is the population size at time t. The theoretical solution is given by $N(t) = 1000e^{0.2t}$.

This question was solved by Adesanya A.A *et al.(2014)*

4.3 Problem 3

Consider the ordinary differential equation:

$y' = -y,\ y(0) = 1,\ 0 \leq x \leq 1,\ h = 0.1,$

Whose exact solution is $y(x) = e^{-x}$.

This problem was solved by James *et al.*(2013)

Table 1. Result for problem 1

x	**Exact**	**Error 1**	**Error 2**	**Error 3**
0.1	1.0050125208594010	1.00153(-12)	1.00658(-12)	1.00876(-12)
0.2	1.0202013400267558	2.03284(-12)	2.05096(-12)	2.05734(-12)
0.3	1.0460278599087169	3.12245(-12)	3.16518(-12)	3.17844(-12)
0.4	1.0832870676749586	4.30141(-12)	4.38440(-12)	4.40854(-12)
0.5	1.1331484530668263	5.60474(-12)	5.74970(-12)	5.78980(-12)
0.6	1.1972173631218102	7.07310(-12)	7.30910(-12)	7.37260(-12)
0.7	1.2776213132048868	8.75490(-12)	9.12110(-12)	9.21825(-12)
0.8	1.3771277643359572	1.07080(-11)	1.12581(-11)	1.14031(-11)
0.9	1.4993025000567668	1.30062(-11)	1.38108(-11)	1.40234(-11)
1.0	1.6487212707001282	1.57369(-10)	1.68940(-10)	1.72019(-10)

Table 2. Result for problem 2

x	**Exact**	**Error 1**	**Error 2**	**Error 3**
0.1	1020.201340026755839	1.3863(13)	6.0274(-13)	1.0096(-13)
0.2	1040.810774192388254	2.8345(-14)	1.4082(-15)	7.2803(-15)
0.3	1061.836546545359676	4.3405(-14)	2.2466(-15)	1.4769(-15)
0.4	1083.287067674958745	5.9060(-14)	3.1183(-15)	2.2559(-15)
0.5	1105.170918075647767	7.5327(-14)	4.0243(-15)	3.0656(-14)
0.6	1127.496851579375734	9.2225(-14)	4.9657(-15)	3.9072(-14)
0.7	1150.273798857227387	1.0977(-13)	5.94343(-15)	4.7816(-14)
0.8	1173.510870991810298	1.2799(-13)	6.9586(-15)	5.6896(-14)
0.9	1197.217363121810294	1.4689(-13)	8.0124(-15)	6.6324(-13)
1.0	1221.402758160169932	1.6651(-13)	9.1059(-14)	7.6110(-16)

Table 3. Result for problem 3

x	**Exact**	**Error 1**	**Error 2**	**Error 3**
0.1	0.9048374180359595	9.489871(13)	1.468703(-14)	1.768585(-14)
0.2	0.8187307530779818	1.648158(-13)	6.973306(-13)	7.541117(-14)
0.3	0.7408182206817178	2.207021(-13)	9.992533(-15)	1.353099(-14)
0.4	0.6703200460356393	2.643932(-13)	3.910785(-15)	1.831505(-14)
0.5	0.6065306597126334	2.977135(-13)	7.895454(-14)	2.206650(-14)
0.6	0.5488116360940265	3.222466(-13)	1.108661(-14)	2.493855(-14)
0.7	0.4965853037914095	3.393681(-13)	1.359916(-13)	2.706456(-14)
0.8	0.4493289641172216	3.507080(-14)	1.553335(-14)	2.856045(-14)
0.9	0.4065696597405991	3.559873(-13)	1.697647(-14)	2.952688(-14)
1.0	0.3678794411714423	3.574096(-13)	1.800445(-14)	3.005104(-14)

4.4 Comparison with Existing Method

Table 4. Error comparison for problem 1 with James *et al.* (2013), $p = 7$

x	Exact	Error James(2013)	Error 1	Error 2	Error 3
0.1	1.0050125208594010	1.6554(-11)	1.00153(-12)	1.00658(-12)	1.00876(-12)
0.2	1.0202013400267558	4.3981(-11)	2.03284(-12)	2.05096(-12)	2.05734(-12)
0.3	1.0460278599087169	7.8451(-11)	3.12245(-12)	3.16518(-12)	3.17844(-12)
0.4	1.0832870676749586	1.2662(-10)	4.30141(-12)	4.38440(-12)	4.40854(-12)
0.5	1.1331484530668263	1.9709(-10)	5.60474(-12)	5.74970(-12)	5.78980(-12)
0.6	1.1972173631218102	3.0180(-10)	7.07310(-12)	7.30910(-12)	7.37260(-12)
0.7	1.2776213132048868	4.5771(-10)	8.75490(-12)	9.12110(-12)	9.21825(-12)
0.8	1.3771277643359572	6.8954(-10)	1.07080(-11)	1.12581(-11)	1.14031(-11)
0.9	1.4993025000567668	1.0336(-09)	1.30062(-11)	1.38108(-11)	1.40234(-11)
1.0	1.6487212707001282	1.5435(-09)	1.57369(-10)	1.68940(-10)	1.72019(-10)

Table 5. Error comparison for problem 2 with Adesanya *et al.(2014)*, $p = 7$

x	Exact	Error Adesanya(2014)	Error 1	Error 2	Error 3
0.1	1020.201340026755839	6.82121(-13)	1.3863(13)	6.0274(-13)	1.0096(-13)
0.2	1040.810774192388254	2.04636(-12)	2.8345(-14)	1.4082(-15)	7.2803(-15)
0.3	1061.836546545359676	2.27373(-13)	4.3405(-14)	2.2466(-15)	1.4769(-15)
0.4	1083.287067674958745	1.13686(-12)	5.9060(-14)	3.1183(-15)	2.2559(-15)
0.5	1105.170918075647767	4.54747(-13)	7.5327(-14)	4.0243(-15)	3.0656(-14)
0.6	1127.496851579375734	2.27373(-13)	9.2225(-14)	4.9657(-15)	3.9072(-14)
0.7	1150.273798857227387	3.18323(-12)	1.0977(-13)	5.94343(-15)	4.7816(-14)
0.8	1173.510870991810298	4.54747(-13)	1.2799(-13)	6.9586(-15)	5.6896(-14)
0.9	1197.217363121810294	9.09494(-13)	1.4689(-13)	8.0124(-15)	6.6324(-13)
1.0	1221.402758160169932	2.04636(-12)	1.6651(-13)	9.1059(-14)	7.6110(-16)

Table 6. Error comparison for problem 3 with James *et al.*(2013), $p = 7$

x	Exact	Error James(2013)	Error 1	Error 2	Error 3
0.1	0.9048374180359595	1.7444(-11)	9.489871(13)	1.468703(-14)	1.768585(-14)
0.2	0.8187307530779818	1.5783(-11)	1.648158(-13)	6.973306(-13)	7.541117(-14)
0.3	0.7408182206817178	1.4281(-11)	2.207021(-13)	9.992533(-15)	1.353099(-14)
0.4	0.6703200460356393	1.2925(-11)	2.643932(-13)	3.910785(-15)	1.831505(-14)
0.5	0.6065306597126334	1.1696(-11)	2.977135(-13)	7.895454(-14)	2.206650(-14)
0.6	0.5488116360940265	1.0580(-11)	3.222466(-13)	1.108661(-14)	2.493855(-14)
0.7	0.4965853037914095	9.5701(-11)	3.393681(-13)	1.359916(-13)	2.706456(-14)
0.8	0.4493289641172216	8.6612(-11)	3.507080(-14)	1.553335(-14)	2.856045(-14)
0.9	0.4065696597405991	7.8371(-11)	3.559873(-13)	1.697647(-14)	2.952688(-14)
1.0	0.3678794411714423	7.0927(-11)	3.574096(-13)	1.800445(-14)	3.005104(-14)

5 Discussion of Results

We considered three numerical examples, Tables $1-3$ showed the numerical results and there respective errors. Table $4-6$ presented the comparison of our method with existing methods, it also showed that case (iii) which is the Radau type gives the best results followed by Case (ii) which is the Lobatto type. The implication is that the farer u and w from v; the better the accuracy of the method.

6 Conclusion

We discussed the development of a class of A-stable Runge Kutta collocation methods with seven internal stages for the solution of first order initial value problems. Three free parameters are considered. Predictor - Corrector approach was adopted to generate results for the solution of IVPs. The basic analysis of the method such as order, consistency,zero stability and convergence was tested on the derived schemes and the mthod was found to be A-Stable. Further work can be done by assigning different values to the free parameters to substantiate our claims above. Comparison were made with some existing works and it showed clearly that our methods performed better in terms of accuracy than some of the existing methods compared with. The method developed are of order seven.

Competing Interests

The authors declare no conflict of interests regarding the publcation of this paper.

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