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On Skew Semi-Invariant Submanifolds of Cosymplectic Manifolds

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In this research, we investigate a new class of submanifold of a cosymplectic manifold, called *"skew semi-invariant submanifold"*, for which sufficient conditions are discussed with aim to state the related integrability of distributions. The differential geometric aspects are treated within the standard scheme, in part it is stated that a manifold with non-trivial invariant distribution is CR-manifold. Moreover, we discussed the some properties of sectional curvature of skew-semi invariant submanifolds of a cosymplectic manifold.

Keywords: Skew semi-invariant submanifolds; cosymplectic manifold; integrability conditions of the distributions; sectional curvature tensor; CR-structure.

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1 Introduction

The geometry of submanifolds in spaces endowed with additional structure is very rich source and interesting topics [1]. The theory of CR-submanifolds which was introduced by Bejancu for almost contact geometry [2] and also for almost complex geometry [3] had a great impact of the theory of submanifolds in these ambient manifolds. Several authors studied and discussed the semi-invariant submanifold in Sasakian manifold and also extended to other ambient spaces *see for example*, Chen et al. [4, 5, 6, 7], Bejancu and N. Papaghuic [8], Mangione [9] and Papaghiuc [10]. In 2001, Shoeb et al. [11] studied $\xi^{\perp}$ -submanifolds of a cosymplectic manifold. Latter on Bejancu also defined and studied a semi-invariant submanifold of a locally product manifold [12]. In 1990, Ximin and Shao [13] have discussed a new class of submanifolds of locally product manifolds, *i.e.,* known as skew semi-invariant submanifolds. Recently, in 2017 Siddiqi et al. [14] studied skew semi-inavriant submanifolds of generalized quasi-Sasakian manifolds. The purpose of the present work is to investigate and discuss the *skew semi-invariant submanifold* of a cosymplectic manifold.

2 Preliminaries

Let $\bar{M}$ be a real n -dimensional smooth manifold equipped with an almost contact metric structure (ϕ, ξ, η, g) [2] where ϕ is $(1,1)$-tensor field, ξ a vector field, η is a [2] 1-form and g an Riemannian metric such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(X, \xi) = \eta(X), \quad g(X, \xi) = 1, \tag{2.2}$$

where I is the identity tensor field and $X, Y \in \chi(\bar{M})$.

$\bar{M}$ is called a cosymplectic manifold if it satisfies ([8])

$$(\bar{\nabla}_X \phi)Y = 0, \tag{2.3}$$

$$\bar{\nabla}_X \xi = 0. \tag{2.4}$$

The almost contact manifold $M(\phi, \xi, \eta, g)$ is said to be normal [2] if

$$N_\phi(X, Y) + 2d\eta(X, Y)\xi = 0,$$

where

$$N_\phi(X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y], \quad X, Y \in (TM)$$

is the Nijenhuis tensor field corresponding to the tensor field ϕ and d denotes the exterior derivative operator . It is know that an almost contact metric structure is cosymplectic if and only if $\bar{\nabla}\eta$ and $\bar{\nabla}\Phi$ vanish, where $\bar{\nabla}$ is the covariant differentiation with respect to g and the fundamental 2-form Φ on $\bar{M}$ is defined by

$$\Phi(X, Y) = g(X, \phi Y). \tag{2.5}$$

If a cosymplectic manifold $\bar{M}$ has a constant ϕ- sectional curvature then it is called a cosympletic space $\bar{M}(c)$. The curvature tensor $\bar{M}$ of such a manifold is defined by ([4])

$$\begin{aligned}\bar{R}(X, Y)Z = \frac{c}{4}[&g(\phi Y, \phi Z)X - g(\phi X, \phi Z)Y + \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y + 2g(X, \phi Y)\phi Z]\end{aligned} \tag{2.6}$$

for $X, Y, Z \in \chi(\bar{M})$.

Let $\bar{M}$ is a cosymplectic manifold and M is a m-dimensional Riemannian submanifold isometrically immersed in $\bar{M}$ and $\bar{\nabla}$ is its Levi-Civita connection. For $p \in M$, the tangent vector $X_p \in T_pM$, we can write

$$\phi X_p = PX_p + QX_p, \tag{2.7}$$

where $PX_p \in T_pM$ is tangent to M and $QX_p \in T^{\perp}{}_pM$ a normal to M. For any two vectors $X_p, Y_p \in T_pM$, we have

$$g(\phi X_p, Y_p) = g(PX_p, Y_p)$$

implies that

$$g(PX_p, Y_p) = g(X_p, PY_p).$$

Therefore P and P^2 are all symmetric operators on the tangent space T_pM. If $\alpha(p)$ is an eigen value of P^2 at $p \in \bar{M}$ then $\alpha(p) \in [0, 1]$ where P^2 is a composition of an isometry and a projection [13].

For each $p \in M$, we set

$$D_p^{\alpha} = \mathcal{K}er(P^2 - \alpha(p)I)$$

where I is an identity transformation on T_pM and $\alpha(p)$ an eigenvalue of P^2 at $p \in M$.

Obviously we see that $D_p^0 = \mathcal{K}erP$, $D_p^1 = \mathcal{K}erQ$ where D_p^1 is the maximal ϕ-invariant subspace of T_pM and D_p^0 is the maximal ϕ -anti invariant subspace of T_pM. If $\alpha_1(p), ...\alpha_k(p)$ are all eigenvalues of P^2 at p then T_pM can be decomposed as the direct sum of the mutually orthogonal eigenspaces *i.e.,* D_p^1 and D_p^0

$$T_pM = D_p^{\alpha_1} \oplus \cdots \oplus D_p^{\alpha_k}.$$

Definition 2.1. [9] A submanifold M of a cosymplectic manifold $\bar{M}$ is said to be a skew semi-invariant submanifold of $\bar{M}$ if there exists an integer k and functions $\alpha_1, \cdots, \alpha_k$ defined on $\bar{M}$ with values in $(0, 1)$ such that

1. $\alpha^1(x), \cdots, \alpha_k(x)$ are distinct eigenvalues of P^2 at each $p \in \bar{M}$ with

$$T_pM = D_p^1 \oplus D_p^0 \oplus D_p^{\alpha_1} \oplus \cdots \oplus D_p^{\alpha_k};$$

2. the dimensions of $D_p^1, D_p^0, D_p^{\alpha_1}, \cdots, D_p^{\alpha_k}$ are independent of $p \in \bar{M}$.

Remark 2.1. (i) From Definition 2.1(2) we can also define P-invariant mutually orthogonal distributions

$$D^{\alpha} = \bigcup_{p \in \bar{M}} D_p^{\alpha}, \qquad \alpha \in \{0, \alpha_1, \cdots, \alpha_k, 1\}$$

on $\bar{M}$ and

$$TM = D^1 \oplus D^0 \oplus D^{\alpha_1} \oplus \cdots \oplus D^{\alpha_k}$$

are differentiable (*see*, [5]).

(ii) If $k = 0$ in Definition (2.1) then it follows that P is a structure of type $f(3, -1)$ on $\bar{M}$ and $\dim(D_p^1) = rank(P_p)$, $\dim(D_p^0)$ are independent of $p \in \bar{M}$ [15].

(iii) If $k = 0$, (1) implies (2) then $\bar{M}$ is called a *semi-invariant $\xi^{\perp}$-submanifold.*

(iv) If $k = 0$ and $D_p^1 = \{0\}$ (*resp.* $D_p^0 = \{0\}$) then $\bar{M}$ becomes an *anti invariant* (*resp. invariant*) *$\xi^{\perp}$-submanifold.*

(v) If $D_p^1 = \{0\} = D_p^0$, $k = 1$ and $\alpha_1^2(x)$ is constant then $\bar{M}$ may be said to be a *θ-slant submanifold* with slant angle $\cos\theta = \alpha_1$.

Example 2.1. *We consider the Euclidean space* ^{9} *and denote its points by* $y = (y^j)$. *Let* $(e_j) = , j = 1, ...9$, *be the natural basis defined by* $y^j = \partial/\partial y^j$. *We define a vector field* ξ *and a* 1*-form* η *by* $\xi = \partial/\partial y^9$ *and* $\eta = dy^9$ *respectively and* ϕ *is* $(1,1)$ *tensor field defined by*

$$\phi e_1 = e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = e_8, \quad \phi e_8 = e_3,$$

$$\phi e_4 = cosht(y)e_5 - sinht(y)e_6,$$
$$\phi e_5 = cosht(y)e_4 + sinht(y)e_7,$$
$$\phi e_6 = -sinht(y)e_4 + cosht(y)e_7$$
$$\phi e_7 = sinht(y)e_5 + cosht(y)e_6, \quad \phi e_9 = 0,$$

where $t : R^9 \to (0, \pi/2)$ *is a smooth function. then it is easy to verify that* R^9 *is an almost contact metric manifold with almost contact structure* (ϕ, ξ, η, g) *with associated metric* g *given by* $g(e_i, e_j) = \delta_{ij}$. *The submanifold*

$$M = \left\{ \left(y^1, ..., y^9\right) \in R^9 \left| y^6, y^7, y^8, y^9 = 0 \right.\right\}$$

of R^9 *is a skew semi-invariant submanifold with*

$$D^1 = Span\left\{e_1, e_2\right\}, \quad D^0 = Span\left\{e_3\right\}, \quad D^\alpha = Span\left\{e_4, e_5\right\}$$

where for $x \in M$ *one has* $\alpha(y) = cosht(y)$.

Here, ∇ denote the induced connection on M, than the Gauss and Weingarten equations are given by the following equations respectively

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \qquad X, Y \in TM, \tag{2.8}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^\perp_X N, \qquad N \in T^\perp M, \tag{2.9}$$

where $\bar{\nabla}$, ∇ and $\nabla^\perp$ are the Riemannian, induced Riemannian and induced normal connection in M, $\bar{M}$ and the normal bundle $T^\perp M$ of $\bar{M}$ *respectively* and A_N the Weingarten endomorphism associated with N also A_N and the second fundamental form h related by the equation

$$g(h(X, Y), N) = g(A_N X, Y). \tag{2.10}$$

Let M be a submanifold of a cosymplectic manifold $\bar{M}$ and $X, Y \in TM$, $N \in T^\perp M$. Using

$$\phi N = BN + CN, \qquad BN \in TM, \ CN \in T^\perp M. \tag{2.11}$$

From

$$\phi(\bar{\nabla}_X Y) = \bar{\nabla}_X \phi Y + (\bar{\nabla}_X \phi)Y,$$

and using (2.7), (2.8), (2.9) ,(2.3) and (2.11) we have

$$P(\nabla_X Y) + Q(\nabla_X Y) + Bh(X, Y) + Ch(X, Y) \tag{2.12}$$
$$= \nabla_X PY + h(X, PY) - A_{QY} X + \nabla^\perp_X QY,$$

for $X, Y \in TM$. Comparing tangential and normal components in (2.12) we obtain

$$P\nabla_X Y = \nabla_X PY - Bh(X, Y) - A_{QY} X, \tag{2.13}$$

$$Q\nabla_X Y = h(X, PY) + \nabla^\perp_X QY - Ch(X, Y), \tag{2.14}$$

for $X, Y \in TM$.

Replace X with Y in (2.13) and (2.14), we get

$$P\nabla_Y X = \nabla_Y PX - Bh(Y, X) - A_{QX} Y, \tag{2.15}$$

$$Q\nabla_Y X = h(Y, PX) + \nabla^{\perp}_Y QX - Ch(Y, X), \tag{2.16}$$

Now, subtracting (2.15) from(2.13) and (2.16) from (2.14), we get

$$P[X, Y] = \nabla_X PY - \nabla_Y PX + A_{QX}Y - A_{QY}X \tag{2.17}$$

$$Q[X, Y] = h(X, PY) - h(PX, Y) + \nabla^{\perp}_Y QX - \nabla^{\perp}_X QY. \tag{2.18}$$

We have following lemmas immediately from (2.17) and (2.18)

Lemma 2.1. *A submanifold M is a skew semi-invariant submanifold of a cosymplectic manifold $\bar{M}$, then the distribution D^0 is integrable if and only if*

$$A_{\phi X}Y = A_{\phi Y}X, \qquad \text{for all } X, Y \in D^0. \tag{2.19}$$

Lemma 2.2. *A submanifold M is a skew semi-invariant submanifold of a cosymplectic manifold $\bar{M}$, then the distribution D^1 is integrable if and only if*

$$h(X, \phi Y) = h(\phi X, Y), \qquad \text{for all } X, Y \in D^1. \tag{2.20}$$

We define the covariant derivatives of P and Q in a manner as follows

$$(\nabla_X P)Y = \nabla_X PY - P\nabla_X Y, \tag{2.21}$$

$$(\nabla_X Q)Y = \nabla^{\perp}_X QY - Q\nabla_X Y, \tag{2.22}$$

for all $X, Y \in TM$. Using (2.13) and (2.14) we have

$$(\nabla_X P)Y = Bh(X, Y) + A_{QY}X, \tag{2.23}$$

$$(\nabla_X Q)Y = Ch(X, Y) - h(X, PY). \tag{2.24}$$

Let D^1 and D^2 be two distributions defined on a manifold M. We say that D^1 is parallel with respect to D^2 if for all $X \in D^2$ and $Y \in D^1$, we have $\nabla_X Y \in D^1$. D^1 is called parallel if for $X \in TM$ and $Y \in D^1$, we have $\nabla_X Y \in D^1$, it is easy to verify that D^1 is parallel if and only if the orthogonal complementary distribution of D^1 is also parallel.

Let M be a submanifold of $\bar{M}$. A distribution D on M is said to be totally geodesic if for all $X, Y \in D$ we have $h(X, Y) = 0$. In this case we say also that M is D totally geodesic. For two distributions D^1 and D^2 defined on M, we say that M is $D^1 - D^2$ mixed totally geodesic if for all $X \in D^1$ and $Y \in D^2$ we have $h(X, Y) = 0$.

Now we have the following lemma

Lemma 2.3. *Let M be a skew semi-invariant submanifold of cosymplectic manifold $\bar{M}$. For any distribution D^α, if*

$$A_N BX = BA_N X, \text{ for all } X \in D^\alpha, \ N \in T^{\perp}M,$$

then M is D^α-D^β-mixed totally geodesic, where $\alpha \neq \beta$.

Proof. From the assumption, we have

$$B^2 A_N X - \alpha A_N X = 0$$

which implies that $A_N X \in D^\alpha$. So for all $Y \in D^\beta$, $N \in T^{\perp}M$, $\alpha \neq \beta$, we have

$$g(A_N X, Y) = g(h(X, Y), N) = 0$$

that is $h(X, Y) = 0$.

Hence $\bar{M}$ is D^α-D^β mixed totally geodesic. □

Now from (2.7) and (2.11) we can obtain

$$CQX_p = -QPX_p, \tag{2.25}$$

$$QBN = N - C^2N, \quad \text{for all } X_p \in T_pM,\ N \in T_p^{\perp}M. \tag{2.26}$$

For (2.25), let$X_p \in T_pM$ in $\phi^2 X_p = X_p$, using (2.7) and (2.11), we get

$$(P^2 + BQ)X_p + (QP + CQ)X_p = X_p$$

from which we get (2.25). Similarly using (2.7) and (2.11) in $\phi^2 N = N - \eta(N)\xi$ for $N \in T^{\perp}M$, we get

$$(PB + BC)N + (C^2 + QB)N = N - \eta(N)\xi,$$

which implies (2.26).

Furthermore for $X_p \in D_p^{\alpha_i}$, $\alpha \in \{\alpha_1, ...\alpha_k\}$, we have

$$C^2QX_p = \alpha_i QX_p. \tag{2.27}$$

Also if $X_p \in D_p^0$ then it is clear that $B^2wX_p = 0$. Thus if X_p is an eigenvector of B^2 corresponding to the eigenvalue $\alpha(p) \neq 1$, QX_p is an eigenvector of C^2 with the same eigenvalue $\alpha(p)$. (1.24) implies that $\alpha(p)$ is an eigenvalue of C^2 if and only if $\gamma(p) = 1 - \alpha(p)$ is an eigenvalue of QB. Since QB and C^2 are symmetric operators on the normal bundle $T^{\perp}M$, their eigenspaces are orthogonal. The dimension of the eigenspace of wB corresponding to the eigenvalue $1 - \alpha(p)$ is equal to the dimension of D_p^{α} if $\alpha(p) \neq 1$. Consequently, we have the following lemma

Lemma 2.4. *A submanifold M is a skew semi-invariant submanifold of a cosymplectic manifold M if and only if the eigenvalues of QB are constant and the eigenspaces of QB have constant dimension.*

3 Skew Semi-invariant Submanifold

Theorem 3.1. *Let M be a submanifold of a cosymplectic manifold $\bar{M}$ if $\nabla P = 0$, then M is a skew semi-invariant submanifold. Furthermore each of the B-invariant distributions D^0, D^1 and D^{α_i}, $1 \leq i \leq k$ are parallels.*

Proof. For a fix $p \in M$ any $Y_p \in D^{\alpha_{i_p}}$ and $X \in TM$. Let Y be the parallel translation of Y_p along with integral curve of X. Since $(\nabla_X P)Y = 0$ and from (2.13) we have

$$\nabla_X(P^2 - \alpha(p)Y) = P^2\nabla_X Y - \alpha(p)\nabla_X Y = 0 \tag{3.1}$$

since $(P^2Y - \alpha(p)Y) = 0$ at p, it is identical to 0 on $\bar{M}$. Thus the eigenvalues of P^2 are constant. Moreover, parallel translation of T_pM along any curve is an isometry which preserves each D^{α}. Thus the dimension of D^{α} is constant and $\bar{M}$ is a skew semi-invariant submanifold.

Now if Y is any vector field in D^{α} then we have $P^2Y = \alpha Y$ (α constant), *i.e*, $P^2\nabla_X Y = \alpha\nabla_X Y$ which implies that D^{α} is parallel. □

Now, we see the vanishing of ∇Q. For $X, Y \in TM$, if $(\nabla_X Q)Y = 0$ then (2.21) yields

$$Ch(X, Y) = h(X, BY) \tag{3.2}$$

In particular, if $Y \in D^{\alpha}$, then ((3.2) implies

$$C^2h(X, Y) = \alpha h(X, Y). \tag{3.3}$$

Consequently we have the following proposition:

Proposition 3.1. *Let M be a skew semi-invariant submanifold of a cosymplectic manifold $\bar{M}$, if $\nabla Q = 0$, then M is D^α-D^β -mixed totally geodesic for all $\alpha \neq \beta$. Moreover, if $X \in D^\alpha$ then either $h(X,X) = 0$ or $h(X,X)$ is an eigenvector of C^2 with eigenvalue α.*

The next lemma is easy to prove so we omit the proof.

Lemma 3.1. *M is a submanifold of a cosymplectivc manifold $\bar{M}$, then $\nabla Q = 0$ if and only if $\nabla_X BN = B\nabla^\perp N$ for all $X \in TM$ and $N \in T^\perp M$.*

Theorem 3.2. *Let M be a submanifold of a cosymplectic manifold $\bar{M}$. If $\nabla w = 0$, then $\bar{M}$ is a skew semi-invariant submanifold.*

Proof. If $TM = D^1$, then we are done. Otherwise, we may find a point $p \in M$ and a vector $X_p \in D_p^\alpha$, $\alpha \neq 1$. Set $N_p = QX_p$, then N_p is an eigenvector of QB with eigenvalue $\mu(p) = 1 - \alpha(p)$. Now, let $Y \in TM$ and N be the translation of N_p in the normal bundle $T^\perp M$ along with integral curve of Y, we have

$$\nabla^\perp_Y(QBN - \mu(p)N) = \nabla^\perp_Y QBN - \mu(p)\nabla^\perp_Y N = Q(\nabla_Y BN) - \mu(p)\nabla^\perp_Y N. \tag{3.4}$$

By Lemma 3.1,

$$\nabla^\perp_Y(QBN - \mu(p)N) = \nabla^\perp_Y QBN - \mu(p)\nabla^\perp_Y N = 0. \tag{3.5}$$

Since $QBN - \mu(p)N = 0$ at p and $QBN - \mu(p)N = 0$ on M. It follows from Lemma 2.4 that M is a skew semi-invariant submanifold. □

Theorem 3.3. *Let M be a skew semi-invariant submanifold of a cosymplectic manifold $\bar{M}$, then the following relation are equivalent:*

1. $(\nabla_X Q)Y - (\nabla_Y W)X = 0$, for all $X, Y \in D^\alpha$,
2. $h(PX, Y) = h(X, PY)$ for all $X, Y \in D^\alpha$,
3. $Q[X,Y] = \nabla^\perp_X QY - \nabla^\perp_Y QX$ for all $X, Y \in D^\alpha$,
4. $A_N PY - PA_N Y$ *is perpendicular to D^α for all $Y \in D^\alpha$ and $N \in T^\perp N$.*

Proof. The proof is trivial, hence we omit it. □

We call P α commutataive if any of the equivalent conditions in the above Lemma holds.

For each P invariant D^α, let $n(\alpha) = dim D^\alpha$. For each D^α we may choose a local orthonormal basis $E^1, ...E^{n(\alpha)}$. Define the D^α mean curvature vector by $H^\alpha = \sum_i^{n(\alpha)} h(E^i, E^i)$, then the mean curvature vector is given by $H = \frac{1}{n}(H^0 + H^1 + H^{\alpha_1} + ...H^{\alpha_k}), n = dim M$.

A skew semi-invaraint submanifold M of a cosymplectic manifold $\bar{M}$ is called D^α minimal if $H^\alpha = 0$ and minimal if $H = 0$.

The equation of Gauss is given by ([5])

$$R(X,Y,Z,W) = \bar{R}(X,Y,Z,W) + g(h(X,W), h(Y,Z)) - g(h(X,Z), h(Y,W)) \tag{3.6}$$

where R and $\bar{R}$ are the curvature of M and $\bar{M}$ respectively.

4 Sectional Curvature of a Skew Semi-invariant Submanifold of a Cosymplectic Space Form $\bar{M}(c)$

Let M be a skew-semi-invariant submanifold of a cosymplectic sapce form $\bar{M}(c)$ is given by

$$\begin{aligned} R(X,Y,Z,W) = \frac{c}{4}[&g(Y,Z)g(X,W) - \eta(Y)\eta(Z)g(X,W) \\ &-g(X,Z)g(Y,W + \eta(X)\eta(Z)g(Y,W) \\ &+\eta(Y)g(X,Z)g(\xi,W) - \eta(X)g(Y,Z)g(\xi,W) \\ &+g(PY,Z)g(PX,W) - g(PX,Z)g(PY,W) \\ &+2g(X,PY)g(PZ,W) + g((h(Y,Z),h(X,W)) \\ &-g(h(X,Z),h(Y,W)) \end{aligned} \tag{4.1}$$

for $X,Y,Z,W \in TM$.

Thus we have

Proposition 4.1. *Let M be a skew semi-invariant submanifold of a cosymplectic space form $\bar{M}(c)$. Then the sectional curvature $K_M(X \wedge Y)$ is given by*

$$\begin{aligned} K_M(X \wedge Y) = \frac{c}{4}&[1 - \eta(X)^2 - \eta(Y)^2 + 3g(X,\phi Y)^2] \\ &+ g(h(Y,Y),h(X,X)) - \|h(X,Y)\|^2 \end{aligned} \tag{4.2}$$

for all orthonormal vectors $X,Y \in TM$.

For any unit vector $X \in D^\alpha$, $\alpha \neq 0$, defined the sectional curvature of $\bar{M}$ and M by

$$\bar{H}_\alpha(X) = K_{\bar{M}}(X \wedge Y), \qquad H_\alpha(X) = K_M(X \wedge Y)$$

respectively, where $Y = \frac{PX}{\sqrt{\alpha}}$. From (4.2) we have

$$H_\alpha(X) = \bar{H}_\alpha(X) = -\frac{1}{\alpha}g(h(X,X),h(\phi X,\phi X)) - \frac{1}{\alpha}\|h(X,\phi X)\|^2. \tag{4.3}$$

Then we have the following proposition

Proposition 4.2. *Let M be a skew semi-invariant submanifold of a cosymplectic manifold $\bar{M}$, if ϕ is a α commutative, $\alpha \neq 0$, then*

$$H_\alpha(X) = \bar{H}_\alpha(X) + \|h(X,X)\|^2 \frac{1}{\alpha}\|h(X,\phi X)\|^2 \tag{4.4}$$

Let $\left\{E^1,....E^{n(\alpha)}\right\}$ and $\left\{F^1,....F^{n(\beta)}\right\}$ be the local orthonormal bases for D^α and D^β, respectively. We define $\alpha-\beta$ sectional curvatures of $\bar{M}$ and M by

$$\bar{\lambda}_{\alpha\beta} = \sum_{i=1}^{n(\alpha)}\sum_{j=1}^{n(\beta)} K_{\bar{M}}(E^i \wedge F^j), \qquad \lambda_{\alpha\beta} = \sum_{i=1}^{n(\alpha)}\sum_{j=1}^{n(\beta)} K_M(E^i \wedge F^j),$$

respectively.

From (4.3) we see that for $\alpha \neq \beta$ we have

$$\lambda_{\alpha\beta} = \bar{\lambda}_{\alpha\beta} + g(H^i,H^j) \tag{4.5}$$

$\lambda_{\alpha\beta} \geq \bar{\lambda}_{\alpha\beta}$ *if* $g(H^i, H^j)$ *is non-negative.*
For $\alpha = \beta$ *we have*

$$\lambda_{\alpha\beta} = \bar{\lambda}_{\alpha\beta} - \sum_{i=1}^{n(\alpha)} \sum_{j=1}^{n(\beta)} \left\| h(E^i \wedge F^j) \right\|^2. \tag{4.6}$$

Using (4.5) *and* (4.6) *we have the following proposition*

Proposition 4.3. *Let* M *be a skew semi-invariant submanifold of a cosymplectic manifold* $\bar{M}$.

1. *If* H^α *is perpendicular to* H^β, $\alpha \neq \beta$, *then* $\lambda_{\alpha\beta} \leq \bar{\lambda}_{\alpha\beta}$, *and the equality holds if and only if* M *is* $D^{\alpha-}D^\beta$ *mixed totally geodesic.*
2. *If* M *is* D^α *minimal, then* $\lambda_{\alpha\beta} \leq \bar{\lambda}_{\alpha\beta}$, *and the equality holds of and only if* M *is*D^α *totally geodesic.*

5 CR-structure

Let $\bar{M}$ *be a differentiable manifold and* $T^c\bar{M}$ *be the complexified tangent bundle to* $\bar{M}$. *A* CR*-structure [2] on* M *is complex sub-bundle* H *of* $T^c\bar{M}$ *such that* $H \cap \bar{H} = \{0\}$ *and* H *is involutive. A manifold endowed with a* CR*-structure is called a* CR*-manifold. It is known that a differentiable manifold* $\bar{M}$ *admits a* CR*-structure [2] if and only if there is a differentiable distribution* $\bar{D}$ *and a* $(1,1)$ *tensor field* P *on* M *such that for all* $X, Y \in \bar{D}$

$$P^2X = -X,$$
$$[P,P](X,Y) \equiv [PX,PY] - [X,Y] - P[PX,Y] - P[X,PY] = 0,$$
$$[PX,PY] - [X,Y] \in \bar{D}.$$

Definition 5.1. A differentiable manifold $\bar{M}$ is said to admit a CR-structure if there is a differentiable distribution $\bar{D}$ and a $(1,1)$ tensor field P on $\bar{M}$ such that for all $X, Y \in \bar{D}$

$$P^2X = X,$$
$$[P,P](X,Y) \equiv [PX,PY] + [X,Y] - P[PX,Y] - P[X,PY] = 0,$$
$$[PX,PY] = [X,Y] \in D.$$

A manifold equipped with a CR-structure is called a CR-manifold.

Lemma 5.1. *An almost contact metric structure* (ϕ, ξ, η, g) *is normal if the Nijenhuis tensor* $[\phi, \phi]$ *of* ϕ *satisfies [8]*

$$[\phi, \phi] + 2d\eta \otimes \xi = 0. \tag{5.1}$$

Now, we prove the following theorem:

Theorem 5.1. *If* $\bar{M}$ *is an skew semi-invariant* $\xi^\perp$*-submanifold of a normal almost contact metric manifold* M *with non-trivial invariant distribution, then* $\bar{M}$ *possesses a* CR*-structure.*

Proof. Since M is normal for $X, Y \in \bar{D}^\perp$, we get $P^2X = -X$ and in view of (4.1), we have

$$0 = [P,P](X,Y) - Q([X,PY] + [PX,Y]),$$

from which it follows that

$$Q([PX,Y] + [X,PY]) = 0,$$

that is $[PX,Y] + [X,PY] \in \bar{D}^1$. Thus

$$[PX,PY] + [X,Y] = P([PX,Y] + [X,PY]) \in \bar{D}^1 \tag{5.2}$$

and hence $(\bar{D}^1, P)$ is a CR-structure on M.

□

Theorem 5.2. *A skew semi-invariant $\xi^{\perp}$-submanifold of a cosymplectic manifold with non-trivial invariant distribution is a CR-manifold.*

Proof. Since every cosymplectic manifold is normal [11], by the Theorem 5.3, the proof is obvious. □

From Theorem 5.3, it is obvious that normality of $\bar{M}$ is a sufficient condition for a skew semi-invariant submanifold with nontrivial invariant distribution to carry a CR-structure. However, this is not neccessary , and now we give an example of skew semi-ivarian

Example 5.1. *We consider the Euclidean space ^{5} and denote its points by $x = (x^i)$. Let $(e_j) = , j = 1, ...5$ be the natural basis defined by $e^j = \partial/\partial x^j$. We define a vector field ξ and a 1-form η by $\xi = \partial/\partial x^5$ and $\eta = dx^5$ respectively. For each $x \in R^5$, and g the canonical metric defined by $g(e_i, e_j) = \delta_i j$, $ij = 1, ...5$, the set $(E_j$ defined by*

$$E_1 = e_1, \quad E_2 = cosh(x^1)e_2 + sinh(x^1)e_3, \quad E_3 = -sinh(x^1)e_2 + cosh(x^1)e_3, \quad E_4 = e_4, \quad E_5 = e_5$$

forms an orthonormal basis.As the point x varies in R^5 the above set of equations defines 5 vector field also denoted by (E_j) and ϕ is $(1,1)$ tensor field defined by

$$\phi(E_1) = E_2, \quad \phi(E_2) = E_1, \quad \phi(E_3) = E_4, \quad \phi(E_4) = E_3 \quad \phi(E_5) = 0.$$

Then (ϕ, ξ, η, g) define an almost contact metric structure on R^5. Since

$$[\phi, \phi](E_1, E_4) + 2d\eta(E_1, E_4)\xi = E_1 \neq 0,$$

the almost contact structure is not normal. The submanifold

$$M = \left\{x \in R^5 : x^4, x^5 = 0\right\}$$

is a skew semi-invariant submnaifold of R^5 with $D^1 = Span\{E_1, E_2\}$ and $D^0 = Span\{E_3\}$ such that (D^1, ϕ) is a CR-structure on M. Moreover, D^1 is not integrable because $[E_1, E_2] = E_3$.

6 Conclusions

We obtain integrability conditions of the distributions on skew semi-invariant submanifold. Moreover, we have discussed a skew semi-invariant $\xi^{\perp}$-submanifold of a Cosymplectic manifold with non-trivial invariant distribution and some relations of sectional curvature tensor form. An example of dimension 5 is given to show that a skew semi-invariant $\xi^{\perp}$ submanifold is a CR-structure on the manifold.

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Competing Interests

Author has declared that no competing interests exist.

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