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Boundedness of Calderón−Zygmund operators and their commutator on Morrey-Herz Spaces with variable exponents

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Abstract: In this paper, the boundedness of Calderón−Zygmund operators is obtained on Morrey-Herz spaces with variable exponents $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$ and several norm inequalities for the commutator generated by Calderón−Zygmund operators, BMO function and Lipschitz function are given.

Keywords: Calderón−Zygmund operators, Morrey-Herz spaces, commutators, variable exponent, BMO spaces, Lipschitz spaces.

MSC: 62D05.

1. Introduction

Let K be a locally integrable function on $\mathbb{R}^n \times \mathbb{R}^n \backslash \{(x,y) : x = y\}$, then we say that K is a standard kernel if there exist $\varepsilon > 0$ and $C > 0$, such that

$$|K(x,y)| \leq C/|x-y|^n, x \neq y;$$
$$|K(x,y) - K(x,w)| \leq C\frac{|y-w|^\varepsilon}{|x-y|^{n+\varepsilon}}, |y-w| \leq \frac{1}{2}|x-y|;$$
$$|K(x,y) - K(z,y)| \leq C\frac{|x-z|^\varepsilon}{|x-y|^{n+\varepsilon}}, |x-z| \leq \frac{1}{2}|x-y|.$$

We say that a linear operator $T : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ is a Calderón−Zygmund operator associated to a standard kernel K if

1. T can be extended to a bounded operator on $L^2(\mathbb{R}^n)$;
2. for all $h \in L^2(\mathbb{R}^n)$ with compact support and almost everywhere $x \notin \operatorname{supp} h$,

$$Th(x) = \int_{\mathbb{R}^n} K(x,y)h(y)dy.$$

Now, suppose that $b \in BOM(\mathbb{R}^n)$ and T be a Calderón−Zygmund operators. The commutator $[b,T]$ generated by b is defined by

$$[b,T]h(x) = b(x)Th(x) - T(bh)(x). \quad (1)$$

In recent decades, the generalized Lebesgue spaces with variable exponent and the corresponding Sobolev spaces with variable exponent have attracted attention of researchers. Due to the fundamental paper [1] by Kováčik and Rákosník appeared in 1991, the theory of these spaces made progress rapidly and these studies have many applications in partial differential equations, fluid dynamics and image restoration [2–5]. One of the main problems on the theory of function spaces is the boundedness of the Hardy-Littlewood maximal operator on Lebesgue spaces with variable exponent. Many researchers [6–9] considered the question of sufficient conditions on the exponent function $p(x)$ to obtained the boundedness of Hardy-Littlewood maximal operators.

Jouné proved that if T is a ε-Calderón$-$Zygmund operator, then T is bounded on $L^p(\mathbb{R}^n)$ [10]. Coifman, Rochberg and Weiss proved that the commutator $[b, T]$ is bounded on $L^p(\mathbb{R}^n)(1 < p < 1)$ [11]. In 1997, Lu [12] showed the commutator $[b, T$ on Herz-Type spaces. In 2006, Cruz-Uribe *et al.*, [13] established the boundedness of some classical operators on variable L^p spaces by applying the theory of weighed norm inequalities and extrapolation.

The Morrey-Herz spaces have been playing a central role in harmonic analysis [14]. The boundedness of some operators and their corresponding characterization of these spaces with variable exponent $p(x)$ were studied widely [15,16]. Recently, Morrey-Herz spaces $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$ and $M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$ with three variable exponents were studied by Wang and Tao [17].

2. Definition of function spaces with variable exponent

In this section we will recall the definition of Lebesgue spaces with variable exponents and the Morrey-Herz spaces with three variable exponents. Let Ω be a measurable set in $\mathbb{R}^n$ with $|\Omega| > 0$.

Definition 1. [11] Let $p(\cdot) : \Omega \to [1, \infty)$ be a measurable function, the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ h\text{is measurable} : \int_\Omega \left(\frac{|h(x)|}{\eta} \right)^{p(x)} dx < \infty\text{for some constant } \eta > 0 \right\}.$$

The space $L^{p(\cdot)}_{Loc}(\Omega)$ is defined by $L^{p(\cdot)}_{Loc}(\Omega) = \{\text{h is measurable} : h \in L^{p(\cdot)}(K) \text{ for all compact } K \subset \Omega\}$. The Lebesgue spaces $L^{p(\cdot)}(\Omega)$ is a Banach spaces with the norm defined by

$$\|h\|_{L^{p(\cdot)}(\Omega)} = \inf\{\eta > 0 : \int_\Omega \left(\frac{|h(x)|}{\eta} \right)^{p(x)} dx \leq 1\},$$

where $p_- = \text{ess inf}\{p(x) : x \in \Omega\}$, $p_+ = \text{ess sup}\{p(x) : x \in \Omega\}$. Then $\mathcal{P}(\Omega)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.

Let M be the Hardy-Littlewood maximal operator. We denote $\mathcal{B}(\Omega)$ to be the set of all function $p(\cdot) \in \mathcal{P}(\Omega)$ such that M is bounded on $L^{p(\cdot)}(\Omega)$.

Let us turn to recall the definition of Herz spaces and Herz-Morrey spaces with variable exponents. We use the following notation;

Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}, C_k = B_k \backslash B_{k-1}, \chi_k = \chi_{C_k}, k \in \mathbb{Z}$.

Definition 2. [17] Let $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n), \alpha(\cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}$ with $\alpha \in L^\infty(\mathbb{R}^n)$ and $0 \leq \lambda < \infty$. The nonhomogeneous Morrey-Herz space with variable exponent $MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$ and homogeneous Morrey-Herz space with variable exponents $M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)$ are defined by

$$MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n) = \left\{ h \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n \backslash \{0\}) : \|h\|_{MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$

and

$$M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n) = \left\{ h \in L^{p(\cdot)}_{Loc}(\mathbb{R}^n \backslash \{0\}) : \|h\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)} < \infty \right\},$$

respectively, where

$$\|h\|_{MK^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)} = \inf \left\{ \eta > 0 : \sup_{k_0 \in z} 2^{-k_0\lambda} \sum_{k=0}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)}|h\chi_k|}{\eta} \right)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leq 1 \right\},$$

$$\|h\|_{M\dot{K}^{\alpha(\cdot),\lambda}_{q(\cdot),p(\cdot)}(\mathbb{R}^n)} = \inf \left\{ \eta > 0 : \sup_{k_0 \in z} 2^{-k_0\lambda} \sum_{k=-\infty}^{k_0} \left\| \left(\frac{2^{k\alpha(\cdot)}|h\chi_k|}{\eta} \right)^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leq 1 \right\}.$$

Remark 1. [17] Let $v \in \mathbb{N}, a_v \geq 0, 1 \leq p_v < \infty$. Then $\sum_{v=0}^{\infty} a_v \leq \left(\sum_{v=0}^{\infty} a_v\right)^{p_*}$, where $p_* = \begin{cases} \min_{v\in\mathbb{N}} p_v, \sum_{v=0}^{\infty} a_v \leq 1, \\ \max_{v\in\mathbb{N}} p_v, \sum_{v=0}^{\infty} a_v > 1. \end{cases}$

Definition 3. [18] For all $0 < \beta \leq 1$, the Lipschitz space $Lip_\beta(\mathbb{R}^n)$ is defined by

$$Lip_\beta(\mathbb{R}^n) = \left\{ h : \|h\|_{Lip_\beta(\mathbb{R}^n)} = \sup_{x,y\in\mathbb{R}^n; x\neq y} \frac{|h(x)-h(y)|}{|x-y|^\beta} < \infty \right\}.$$

3. Properties and lemmas of variable exponent

Proposition 1. *[19] If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, then*

$$|p(x)-p(y)| \leq \frac{-C}{Log(|x-y|)}, \text{ if } |x-y| \leq 1/2,$$
$$|p(x)-p(y)| \leq \frac{C}{Log(e+|x|)}, \text{ if } |y| \geq |x|.$$

Lemma 1. *[1] Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $h \in L^{p(\cdot)}$ and $g \in L^{p'(\cdot)}$, then hg is integrable on $\mathbb{R}^n$ and*

$$\int_{\mathbb{R}^n} |h(x)g(x)|dx \leq C_p \|h\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)},$$

where $C_p = 1 + \frac{1}{p_-} - \frac{1}{p_+}$.

Lemma 2. *[1] Suppose that $p(\cdot), p_1(\cdot), p_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and for any $h \in L^{p_1(\cdot)}(\mathbb{R}^n)$, $g \in L^{p_2(\cdot)}(\mathbb{R}^n)$, when $\frac{1}{p(\cdot)} = \frac{1}{p_2(\cdot)} + \frac{1}{p_1(\cdot)}$, we get*

$$\|h(x)g(x)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|h\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)},$$

where $C_{p_1,p_2} = [1 + \frac{1}{p_{1-}} - \frac{1}{p_{1+}}]^{\frac{1}{p_-}}$.

Lemma 3. *[20] Let $b \in BMO(\mathbb{R}^n)$ and $i, j \in \mathbb{Z}$ with $i < j$, then*

1. $C^{-1}\|b\|_{BMO(\mathbb{R}^n)} \leq \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|(b-b_B)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_{BMO(\mathbb{R}^n)}$;
2. $\|(b-b_{B_i})\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C(j-i)\|b\|_{BMO(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{q(\cdot)}(\mathbb{R}^n)}$.

Lemma 4. *[21,22] Let $p_u(\cdot) \in \mathfrak{B}(\mathbb{R}^n)(u = 1,2)$, then there exist constants $0 < \delta_{u1}, \delta_{u2} < 1$ and $C > 0$ such that for all balls $B \subset \mathbb{R}^n$ and all measurable subset $R \subset B$, we have*

$$\frac{\|\chi_B\|_{L^{p_u(\cdot)}(\mathbb{R}^n)}}{\|\chi_R\|_{L^{p_u(\cdot)}(\mathbb{R}^n)}} \leq C\frac{|B|}{|R|}, \frac{\|\chi_R\|_{L^{p_u(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_u(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|R|}{|B|}\right)^{\delta_{u2}}, \frac{\|\chi_R\|_{L^{p'_u(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p'_u(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|R|}{|B|}\right)^{\delta_{u1}}.$$

Lemma 5. *[11] If $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, there exist a constant $C > 0$ such that for any balls B in $\mathbb{R}^n$, we have*

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leq C.$$

Lemma 6. *[11] Suppose $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $h \in L^{p(\cdot)q(\cdot)}$, then*

$$\min\left(\|h\|^{q_+}_{L^{p(\cdot)q(\cdot)}}, \|h\|^{q_-}_{L^{p(\cdot)q(\cdot)}}\right) \leq \||h|^{q(\cdot)}\|_{L^{p(\cdot)}} \leq \max\left(\|h\|^{q_+}_{L^{p(\cdot)q(\cdot)}}, \|h\|^{q_-}_{L^{p(\cdot)q(\cdot)}}\right).$$

Proposition 2. *[11] Let I_β be a fractional integrals operator $p_1(\cdot), p_2(\cdot) \in \mathcal{B}(\mathbb{R}^n)$ and $0 < \beta < n/(p_1)_+$. If $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\beta}{n}$, then we have*

$$\|I_\beta h\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \leq C\|h\|_{L^{p_1(\cdot)}(\mathbb{R}^n)},$$

for all $h \in L^{p_1(\cdot)}$.

Lemma 7. *[11] Suppose that $[b,T]$ as defined in (1) and $p_1(\cdot),p_2(\cdot)\in\mathcal{B}(\mathbb{R}^n)$. If $b\in Lip_\beta(\mathbb{R}^n)$ $(0<\beta<n/(p_1)_+)$ and $\frac{1}{p_1(x)}-\frac{1}{p_2(x)}=\frac{\beta}{n}$, then $[b,T]$ is bounded from $L^{p_2(\cdot)}(\mathbb{R}^n)$ in to $L^{p_1(\cdot)}(\mathbb{R}^n)$.*

Proof. Set $b\in Lip_\beta(\mathbb{R}^n)(0<\beta<1)$, then

$$\begin{aligned}|[b,T](h)(x)|&\le\int_{\mathbb{R}^n}|(b(x)-b(y))K(x,y)h(y)|dy\\&\le\int_{\mathbb{R}^n}|(b(x)-b(y))\frac{C}{|x-y|^n}h(y)|dy\\&\le C\|b\|_{Lip_\beta(\mathbb{R}^n)}\int_{\mathbb{R}^n}\frac{|h(y)|}{|x-y|^{n-\beta}}dy.\end{aligned}$$

Notice that $0<\beta<n/(p_1)_+$ so by applying Proposition 2, therefore

$$\|[b,T](h)\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\le C\|b\|_{Lip_\beta(\mathbb{R}^n)}\|I_\beta(|h|)\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\le C\|b\|_{Lip_\beta(\mathbb{R}^n)}\|h\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}.$$

□

4. Main result and proof

Theorem 1. *Suppose that $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, $q_1(\cdot),q_2(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ with $(q_2)_-\ge(q_1)_+$. If $\lambda_1(q_2)_+=\lambda_2(q_1)_-$, $\lambda_1/(q_1)_--n\delta_{12}<\alpha_+<\lambda_1/(q_1)_-+n\delta_{11}$ with δ_{11},δ_{12} as in Lemma 4, then the operator T is bounded from $MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)$ to $MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)$.*

Proof. Let $h\in MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)$. Write

$$h(x)=\sum_{j=0}^{\infty}h(x)\chi_j(x)\triangleq\sum_{j=0}^{\infty}h_j(x).$$

By the Definition 2, we get

$$\|T(h)\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)}=\inf\left\{\eta>0:\sup_{k_0\in z}2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|T(h)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\le 1\right\}.$$

For any $k_0\in\mathbb{Z}$, we have

$$\begin{aligned}2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|T(h)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}&\le 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{\infty}T(h_j)\chi_k\right|}{\sum_{i=1}^{3}\eta_{1i}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\\&\le 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{k-2}T(h_j)\chi_k\right|}{\eta_{11}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}+2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k-1}^{k+1}T(h_j)\chi_k\right|}{\eta_{12}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\\&\quad+2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k+2}^{\infty}T(h_j)\chi_k\right|}{\eta_{13}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}.\end{aligned}$$

Let

$$\eta_{11}=\left\|\sum_{j=0}^{k-2}T(h_j)\right\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)}=\inf\left\{\eta>0:\sup_{k_0\in z}2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{k-2}T(h_j)\chi_k\right|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\leq 1\right\}$$

$$\eta_{12}=\left\|\sum_{j=k-1}^{k+1}T(h_j)\right\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)}=\inf\left\{\eta>0:\sup_{k_0\in z}2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k-1}^{k+1}T(h_j)\chi_k\right|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\leq 1\right\}$$

$$\eta_{13}=\left\|\sum_{j=k+2}^{\infty}T(h_j)\right\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)}=\inf\left\{\eta>0:\sup_{k_0\in z}2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k+2}^{\infty}T(h_j)\chi_k\right|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\leq 1\right\}$$

and

$$\eta=\sum_{i=1}^{3}\eta_{1i}.$$

Thus, we have

$$2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|T(h)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\leq C.$$

This implies that

$$\|T(h)\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)}\leq C\eta=C\sum_{i=1}^{3}\eta_{1i}. \tag{2}$$

Hence, it suffices to prove

$$\eta_{11},\eta_{12},\eta_{13}\leq C\eta_{10}\leq C\|h\|_{MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)}.$$

Denote $\eta_{10}\leq C\|h\|_{MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)}$

Step 1. We first estimate η_{12}. By Lemma 6 and the T-boundedness in $L^{p(\cdot)}$ (see [10]), we conclude that

$$\begin{aligned}2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k-1}^{k+1}T(h_j)\chi_k\right|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}&\leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=k-1}^{k+1}T(h_j)\chi_k|}{\eta_{10}}\right\|_{L^{p(\cdot)}}^{(q_2^1)_k}\\ \leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(\sum_{j=k-1}^{k+1}\left\|\frac{2^{(k-j)\alpha_+}2^{j\alpha_+}|T(h_j)\chi_k|}{\eta_{10}}\right\|_{L^{p(\cdot)}}\right)^{(q_2^1)_k}&\leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(\sum_{j=k-1}^{k+1}\left\|\frac{2^{j\alpha_+}|h_j|}{\eta_{10}}\right\|_{L^{p(\cdot)}}\right)^{(q_2^1)_k},\end{aligned} \tag{3}$$

where

$$(q_2^1)k=\begin{cases}(q_2)_-, & \left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=k-1}^{k+1}T(h_j)\chi_k|}{\eta_{10}}\right\|_{L^{p(\cdot)}}\leq 1,\\ (q_2)_+, & \left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=k-1}^{k+1}T(h_j)\chi_k|}{\eta_{10}}\right\|_{L^{p(\cdot)}}>1.\end{cases}$$

By applying Lemma 6 in (3) and assuming that $\lambda_1(q_2)_+ = \lambda_2(q_1)_-$, we have

$$2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=k-1}^{k+1}T(h_j)\chi_k|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(\sum_{j=k-1}^{k+1}\left\|\frac{2^{j\alpha_+}|h_j|}{\eta_{10}}\right\|_{L^{p(\cdot)}}\right)^{(q_2^1)_k}$$

$$\leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(\left\|\frac{2^{k\alpha_+}|h\chi_k|}{\eta_{10}}\right\|_{L^{p(\cdot)}}\right)^{(q_2^1)_k} \leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha_+}|h\chi_k|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}^{\frac{(q_2^1)_k}{(q_1^1)_k}}$$

$$\leq \sum_{k=0}^{k_0}\left\{2^{-k_0\lambda_1}\left\|\left(\frac{2^{k\alpha_+}|h\chi_k|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}\right\}^{\frac{(q_2^1)_k}{(q_1^1)_k}}$$

where

$$(q_1^1)_k = \begin{cases} (q_1)_-, & \left\|\frac{2^{k\alpha_+}|h\chi_k|}{\eta_{10}}\right\|_{L^{p(\cdot)}} \leq 1, \\ (q_1)_+, & \left\|\frac{2^{k\alpha_+}|h\chi_k|}{\eta_{10}}\right\|_{L^{p(\cdot)}} > 1. \end{cases}$$

Since $h \in MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)$, it is easy to see that

$$2^{-k_0\lambda_1}\left\|\left(\frac{2^{k\alpha_+}|h\chi_k|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}} \leq 1.$$

From above, with $(q_1)_+ \leq (q_2)_-$, we get the following inequality

$$2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=k-1}^{k+1}T(h_j)\chi_k|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq C\sum_{k=0}^{k_0}2^{-k_0\lambda_1}\left\|\left(\frac{2^{k\alpha_+}|h\chi_k|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}} \leq C.$$

These imply that

$$\eta_{12} \leq C\eta_{10} \leq C\|h\|_{MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)}.$$

Step 2. Let us turn to estimate η_{12}. For each $k \in \mathbb{Z}, j \leq k-2$ and a.e. $x \in R_k$, applying the generalized Hölder inequality, we have

$$|Th_j(x)| \leq \int_{R_{k-2}}|K(x,y)||h_j(y)|dy \leq C2^{-kn}\int_{R_j}|h_j(y)|dy \leq C2^{-kn}\|h\|_{L^1(\mathbb{R}^n)}.$$

By Lemma 6, we have

$$2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}T(h_j)\chi_k|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}T(h_j)\chi_k|}{\eta_{10}}\right\|_{L^{p(\cdot)}}^{(q_2^2)_k}$$

$$\leq C2^{-k_\circ\lambda_2}\sum_{k=0}^{k_\circ}\left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}2^{-kn}\|h_j\|_{L^1(\mathbb{R}^n)}\chi_k|}{\eta_{10}}\right\|_{L^{p(\cdot)}}^{(q_2^2)_k} \leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\{2^{k\alpha_+}\sum_{j=0}^{k-2}2^{-kn}\left\|\frac{h_j}{\eta_{10}}\right\|_{L^1(\mathbb{R}^n)}\|\chi_k\|_{L^{p(\cdot)}}\right\}^{(q_2^2)_k},$$

where

$$(q_2^2)_k=\begin{cases}(q_2)_-, & \left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}T(h_j)\chi_k|}{\eta_{10}}\right\|_{L^{p(\cdot)}}\le 1,\\ (q_2)_+, & \left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}T(h_j)\chi_k|}{\eta_{10}}\right\|_{L^{p(\cdot)}}>1.\end{cases}$$

By Lemmas 4 and 5, we have

$$\begin{aligned}
&2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}T(h_j)\chi_k|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\\
&\le C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\{2^{k\alpha_+}\sum_{j=0}^{k-2}2^{-kn}\times\left\|\frac{h_j}{\eta_{10}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}}\|\chi_k\|_{L^{p(\cdot)}}\right\}^{(q_2^2)_k}\\
&\le C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\{2^{k\alpha_+}\sum_{j=0}^{k-2}2^{-kn}\times\left\|\frac{h_j}{\eta_{10}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}}}{\|\chi_k\|_{L^{p'(\cdot)}}}|B_k|\right\}^{(q_2^2)_k}\\
&\le C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\{2^{k\alpha_+}\sum_{j=0}^{k-2}2^{(j-k)n\delta_{11}}2^{-j\alpha_+}\left\|\frac{2^{j\alpha_+}h\chi_j}{\eta_{10}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right\}^{(q_2^2)_k}\\
&\le C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\{\sum_{j=0}^{k-2}2^{(k-j)(\alpha_+-n\delta_{11})}\left\|\frac{2^{j\alpha_+}h\chi_j}{\eta_{10}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right\}^{(q_2^2)_k},
\end{aligned}\tag{4}$$

Applying Lemma 6 on (4), we obtain

$$\begin{aligned}
&2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}T(h_j)\chi_k|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\le C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\{\sum_{j=0}^{k-2}2^{(k-j)(\alpha_+-n\delta_{11})}\left\|\left(\frac{2^{j\alpha_+}h\chi_j}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}^{\frac{1}{(q_1^2)_j}}\right\}^{(q_2^2)_k}\\
&\le C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}2^{(k-k)\lambda_2}\left\{\sum_{j=0}^{k-2}2^{(k-j)(\alpha_+-n\delta_{11})}\times\left(2^{j\lambda_1}2^{-j\lambda_1}\sum_{\ell=0}^{j}\left\|\left(\frac{2^{\ell\alpha_+}h\chi_j}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}\right)^{\frac{1}{(q_1^2)_j}}\right\}^{(q_2^2)_k}\\
&\le C\sum_{k=0}^{k_0}2^{(k-k_0)\lambda_2}\left\{\sum_{j=0}^{k-2}2^{(k-j)(\alpha_+-n\delta_{11}-\lambda_1/(q_1)_-)}\times\left(2^{-j\lambda_1}\sum_{\ell=0}^{j}\left\|\left(\frac{2^{\ell\alpha_+}h\chi_j}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}\right)^{\frac{1}{(q_1^2)_j}}\right\}^{(q_2^2)_k},
\end{aligned}$$

where

$$(q_1^2)_j=\begin{cases}(q_1)_-, & \left\|\frac{2^{j\alpha_+}|h\chi_j|}{\eta_{10}}\right\|_{L^{p(\cdot)}}\le 1,\\ (q_1)_+, & \left\|\frac{2^{j\alpha_+}|h\chi_j|}{\eta_{10}}\right\|_{L^{p(\cdot)}}>1.\end{cases}$$

Noting that $h\in MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)$ and $\alpha_+<n\delta_{11}+\lambda_1/(q_1)_-$, so we get

$$2^{-k_\circ\lambda_2}\sum_{k=0}^{k_\circ}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}T(h_j)\chi_k|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\le C\sum_{k=0}^{k_\circ}2^{(k-k_\circ)\lambda_2}\left(\sum_{j=0}^{k-2}2^{(k-j)(\alpha_+-n\delta_{11}-\lambda_1/(q_1)_-)}\right)^{(q_2^2)_k}\le C.$$

This implies that
$$\eta_{12} \leq C\eta_{10} \leq C\|h\|_{MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)}.$$

Step 3. Finally, we consider η_{13}. For each $j \geq k+2$ and $x \in R_k, y \in R_j$. By the similar argument in Step 2, we obtain that
$$|Th_j(x)| \leq C2^{-jn}\|h\|_{L^1(\mathbb{R}^n)},$$

and
$$2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=k+2}^{\infty}T(h_j)\chi_k|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=k+2}^{\infty}T(h_j)\chi_k|}{\eta_{10}}\right\|_{L^{p(\cdot)}}^{(q_2^3)_k}$$
$$\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=k+2}^{\infty}2^{-jn}\|h_j\|_{L^1(\mathbb{R}^n)}\chi_k|}{\eta_{10}}\right\|_{L^{p(\cdot)}}^{(q_2^3)_k} \leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\{2^{k\alpha_+}\sum_{j=k+2}^{\infty}2^{-jn}\left\|\frac{h_j}{\eta_{10}}\right\|_{L^1(\mathbb{R}^n)}\|\chi_k\|_{L^{p(\cdot)}}\right\}^{(q_2^3)_k},$$

where
$$(q_2^3)k = \begin{cases} (q_2)_-, & \left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=k+2}^{\infty}T(h_j)\chi_k|}{\eta_{10}}\right\|_{L^{p(\cdot)}} \leq 1, \\ (q_2)_+, & \left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=k+2}^{\infty}T(h_j)\chi_k|}{\eta_{10}}\right\|_{L^{p(\cdot)}} > 1. \end{cases}$$

So, by Lemmas 4, 5 and 6, we have
$$\begin{aligned}
&2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=k+2}^{\infty}T(h_j)\chi_k|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\{2^{k\alpha_+}\sum_{j=k+2}^{\infty}2^{-jn}\times\left\|\frac{h_j}{\eta_{10}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}}\|\chi_k\|_{L^{p(\cdot)}}\right\}^{(q_2^3)_k}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\{2^{k\alpha_+}\sum_{j=k+2}^{\infty}2^{-jn}\times\left\|\frac{h_j}{\eta_{10}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_k\|_{L^{p(\cdot)}}}{\|\chi_{B_j}\|_{L^{p(\cdot)}}}|B_j|\right\}^{(q_2^3)_k}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\{2^{k\alpha_+}\sum_{j=k+2}^{\infty}2^{(k-j)n\delta_{12}}2^{-j\alpha_+}\left\|\frac{2^{j\alpha_+}h\chi_j}{\eta_{10}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\right\}^{(q_2^3)_k}\\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\{\sum_{j=k+2}^{\infty}2^{(k-j)(\alpha_++n\delta_{12})}\left\|\left(\frac{2^{j\alpha_+}h\chi_j}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}^{\frac{1}{(q_1^3)_j}}\right\}^{(q_2^3)_k},
\end{aligned}$$

where
$$(q_1^3)_j = \begin{cases} (q_1)_-, & \left\|\frac{2^{j\alpha_+}|h\chi_j|}{\eta_{10}}\right\|_{L^{p(\cdot)}} \leq 1, \\ (q_1)_+, & \left\|\frac{2^{j\alpha_+}|h\chi_j|}{\eta_{10}}\right\|_{L^{p(\cdot)}} > 1. \end{cases}$$

Hence, $h \in MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)$ and $-n\delta_{12} + \lambda_1/(q_1)_- < \alpha_+$, so we get

$$2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=k+2}^{\infty}T(h_j)\chi_k|}{\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\{\sum_{j=k+2}^{\infty}2^{(k-j)(\alpha_+ + n\delta_{12})}\left\|\left(\frac{2^{j\alpha_+}h\chi_j}{\eta_{10}}\right)^{q_1(\cdot)}\right\|^{\frac{1}{(q_1^3)_j}}_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}\right\}^{(q_2^3)_k}$$

$$\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}2^{-k_0\lambda_2}2^{(k-k)\lambda_2}\left\{\sum_{j=k+2}^{\infty}2^{(k-j)(\alpha_+ + n\delta_{12})}\times\left(2^{j\lambda_1}2^{-j\lambda_1}\sum_{\ell=0}^{j}\left\|\left(\frac{2^{\ell\alpha_+}h\chi_j}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}\right)^{\frac{1}{(q_1^3)_j}}\right\}^{(q_2^3)_k}$$

$$\leq C\sum_{k=0}^{k_0}2^{(k-k_0)\lambda_2}\left\{\sum_{j=k+2}^{\infty}2^{(k-j)(\alpha_+ + n\delta_{12}-\lambda_1/(q_1)_-)}\times\left(2^{-j\lambda_1}\sum_{\ell=0}^{j}\left\|\left(\frac{2^{\ell\alpha_+}h\chi_j}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}\right)^{\frac{1}{(q_1^3)_j}}\right\}^{(q_2^3)_k}$$

$$\leq C\sum_{k=0}^{k_0}2^{(k-k_0)\lambda_2}\left(\sum_{j=k+2}^{\infty}2^{(k-j)(\alpha_+ + n\delta_{12}-\lambda_1/(q_1)_-)}\right)^{(q_2^3)_k} \leq C.$$

Hence

$$\eta_{13} \leq C\eta_{10} \leq C\|h\|_{MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)}.$$

This completes the proof Theorem □

Theorem 2. *Suppose $b \in BMO(\mathbb{R}^n)$. Further suppose $p(\cdot) \in \mathcal{B}(\mathbb{R}^n)$, $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $(q_2)_- \geq (q_1)_+$. If $\lambda_1(q_2)_+ = \lambda_2(q_1)_-$, $\lambda_1/(q_1)_- - n\delta_{12} < \alpha_+ < \lambda_1/(q_1)_- + n\delta_{11}$ with δ_{11}, δ_{12} as in Lemma 4, then the commutator $[b, T]$ is bounded from $MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)$ to $MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)$.*

Proof. Let $b \in BMO(\mathbb{R}^n)$, and $h \in MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)$. We write

$$h(x) = \sum_{j=0}^{\infty}h(x)\chi_j(x) \triangleq \sum_{j=0}^{\infty}h_j(x).$$

By the Definition 2, we have

$$\|[b,T](h)\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)} = \inf\left\{\eta > 0 : \sup_{k_\circ\in z}2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|[b,T](h)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 1\right\}.$$

Let

$$\eta_{21} = \left\|\sum_{j=0}^{k-2}[b,T](h_j)\right\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)} = \inf\left\{\eta > 0 : \sup_{k_\circ\in z}2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{k-2}[b,T](h_j)\chi_k\right|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 1\right\},$$

$$\eta_{22} = \left\|\sum_{j=k-1}^{k+1}[b,T](h_j)\right\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)} = \inf\left\{\eta > 0 : \sup_{k_\circ\in z}2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k-1}^{k+1}[b,T](h_j)\chi_k\right|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 1\right\},$$

$$\eta_{23} = \left\|\sum_{j=k+2}^{\infty}[b,T](h_j)\right\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)} = \inf\left\{\eta > 0 : \sup_{k_\circ\in z}2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k+2}^{\infty}[b,T](h_j)\chi_k\right|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 1\right\}.$$

Then, for any $k_0 \in \mathbb{Z}$, we deduce that

$$2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|[b,T](h)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}$$

$$\leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{\infty}[b,T](h_j)\chi_k\right|}{\sum_{i=1}^{3}\eta_{2i}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=0}^{k-2}[b,T](h_j)\chi_k\right|}{\eta_{21}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}$$

$$+2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k-1}^{k+1}[b,T](h_j)\chi_k\right|}{\eta_{22}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} +2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k+2}^{\infty}[b,T](h_j)\chi_k\right|}{\eta_{23}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}},$$

and

$$\eta=\sum_{i=1}^{3}\eta_{2i}.$$

This implies that

$$\|[b,T](h)\|_{MK^{\alpha(\cdot),\lambda_2}_{q_2(\cdot),p(\cdot)}(\mathbb{R}^n)}\leq C\eta=C\sum_{i=1}^{3}\eta_{2i}.$$

Hence, we only need to estimate

$$\eta_{21},\eta_{22}\text{and}\eta_{23}\leq C\|b\|_*\|h\|_{MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)}.$$

Denote $\eta_{10}\leq C\|h\|_{MK^{\alpha(\cdot),\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)}$.

Step 1. We estimate η_{22}. By the boundedness of commutator $[b,T]$ on $L^{p(\cdot)}(\mathbb{R}^n)$, together with Lemma 6, it follows

$$2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}\left|\sum_{j=k-1}^{k+1}[b,T](h_j)\chi_k\right|}{\eta_{10}\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_\circ}\left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=k-1}^{k+1}[b,T](h_j)\chi_k|}{\eta_{10}\|b\|_*}\right\|_{L^{p(\cdot)}}^{(q_2^1)_k}$$

$$\leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_\circ}\left(\sum_{j=k-1}^{k+1}\left\|\frac{2^{(k-j)\alpha_+}2^{j\alpha_+}|[b,T](h_j)\chi_k|}{\eta_{10}\|b\|_*}\right\|_{L^{p(\cdot)}}\right)^{(q_2^1)_k} \leq 2^{-k_0\lambda_2}\sum_{k=0}^{k_\circ}\left(\sum_{j=k-1}^{k+1}\left\|\frac{2^{j\alpha_+}|h_j|}{\eta_{10}}\right\|_{L^{p(\cdot)}}\right)^{(q_2^1)_k},$$

where

$$(q_2^1)k=\begin{cases}(q_2)_-, & \left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=k-1}^{k+1}[b,T](h_j)\chi_k|}{\eta_{10}\|b\|_*}\right\|_{L^{p(\cdot)}}\leq 1,\\ (q_2)_+, & \left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=k-1}^{k+1}[b,T](h_j)\chi_k|}{\eta_{10}\|b\|_*}\right\|_{L^{p(\cdot)}}>1.\end{cases}$$

Therefore, since $h\in MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)$, we can obtain

$$2^{-k_0\lambda_1}\left\|\left(\frac{2^{k\alpha_+}|h\chi_k|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}\leq 1.$$

From this, and by Lemma 6, if $(q_1)_+ \le (q_2)_-$ and $\lambda_1(q_2)_+ = \lambda_2(q_1)_-$, then we get

$$2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=k-1}^{k+1}[b,T](h_j)\chi_k|}{\eta_{10}\|b\|_*}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \le 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(\sum_{j=k-1}^{k+1}\left\|\frac{2^{j\alpha_+}|h_j|}{\eta_{10}}\right\|_{L^{p(\cdot)}}\right)^{(q_2^1)_k}$$

$$\le 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(\left\|\frac{2^{k\alpha_+}|h\chi_k|}{\eta_{10}}\right\|_{L^{p(\cdot)}}\right)^{(q_2^1)_k} \le 2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha_+}|h\chi_k|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}^{\frac{(q_2^1)_k}{(q_1^1)_k}}$$

$$\le \sum_{k=0}^{k_0}\left\{2^{-k_0\lambda_1}\left\|\left(\frac{2^{k\alpha_+}|h\chi_k|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}\right\}^{\frac{(q_2^1)_k}{(q_1^1)_k}}$$

where

$$(q_1^1)_k = \begin{cases} (q_1)_-, & \left\|\frac{2^{k\alpha_+}|h\chi_k|}{\eta_{20}}\right\|_{L^{p(\cdot)}} \le 1, \\ (q_1)_+, & \left\|\frac{2^{k\alpha_+}|h\chi_k|}{\eta_{20}}\right\|_{L^{p(\cdot)}} > 1. \end{cases}$$

This implies

$$\eta_{21} \le C\|b\|_*\eta_{10} \le C\|b\|_*\|h\|_{MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)}.$$

Step 2. Next we estimate η_{22}. Let $x \in R_k, y \in R_j$ and $j \le k-2$ then $2|y| < |x|$ and applying the generalized Hölder's inequality, we have

$$\begin{aligned}|[b,T]h_j(x)| &\le \int_{R_j}|K(x,y)||b(x)-b(y)||h_j(y)|dy \le C2^{-nk}\int_{R_j}|b(x)-b(y)||h_j(y)|dy \\ &\le C2^{-nk}\left[|b(x)-b_{B_j}|\int_{R_j}|h_j(y)|dy + \int_{R_j}|b(y)-b_{B_j}||h_j(y)|dy\right] \\ &\le C2^{-nk}\left[|b(x)-b_{B_j}|\|h\|_{L^1(\mathbb{R}^n)} + \|(b-b_{B_j})h_j\|_{L^1(\mathbb{R}^n)}\right].\end{aligned}$$

Therefore, by Lemma 6, we have

$$2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}[b,T](h_j)\chi_k|}{\|b\|_*\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \le C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}[b,T](h_j)\chi_k|}{\|b\|_*\eta_{10}}\right\|_{L^{p(\cdot)}}^{(q_2^2)_k}$$

$$\le C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}2^{(j-k)\varepsilon}2^{-nk}|b(x)-b_{B_j}|\|h\|_{L^1(\mathbb{R}^n)}\chi_k|}{\|b\|_*\eta_{10}}\right\|_{L^{p(\cdot)}}^{(q_2^2)_k}$$

$$+ C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}2^{-nk}\|(b-b_{B_j})h_j\|_{L^1(\mathbb{R}^n)}\chi_k|}{\|b\|_*\eta_{10}}\right\|_{L^{p(\cdot)}}^{(q_2^2)_k}$$

$$\le C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(2^{k\alpha_+}\sum_{j=0}^{k-2}2^{-nk}\left\|\frac{|(b-b_j)h_j|}{\|b\|_*\eta_{10}}\right\|_{L^1(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{p(\cdot)}}\right)^{(q_2^2)_k}$$

$$+ C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(2^{k\alpha_+}\sum_{j=0}^{k-2}2^{-nk}\left\|\frac{|h_j|}{\eta_{10}}\right\|_{L^1(\mathbb{R}^n)}\|b\|_*^{-1}\|(b-b_j)\chi_{B_k}\|_{L^{p(\cdot)}}\right)^{(q_2^2)_k},$$

where

$$(q_2^2)_k = \begin{cases} (q_2)_-, & \left\| \frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}[b,T](h_j)\chi_k|}{\eta_{10}\|b\|_*} \right\|_{L^{p(\cdot)}} \leq 1, \\ (q_2)_+, & \left\| \frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}[b,T](h_j)\chi_k|}{\eta_{10}\|b\|_*} \right\|_{L^{p(\cdot)}} > 1. \end{cases}$$

By applying Lemmas 3 and 6, we get that

$$\begin{aligned}
&2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\| \left(\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}[b,T](h_j)\chi_k|}{\|b\|_*\eta_{10}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(2^{k\alpha_+}\sum_{j=0}^{k-2}2^{-nk}\left\|\frac{|h_j|}{\eta_{10}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\left\|\frac{|(b-b_j)\chi_{B_j}|}{\|b\|_*}\right\|_{L^{p'(\cdot)}(\mathbb{R}^n)}\|\chi_{B_k}\|_{L^{p(\cdot)}} \right)^{(q_2^2)_k} \\
&+C2^{-k_\circ\lambda_2}\sum_{k=0}^{k_\circ}\left(2^{k\alpha_+}\sum_{j=0}^{k-2}2^{-nk}\left\|\frac{|h_j|}{\eta_{10}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p'(\cdot)}(\mathbb{R}^n)}(k-j)\|\chi_{B_k}\|_{L^{p(\cdot)}} \right)^{(q_2^2)_k} \\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(2^{k\alpha_+}\sum_{j=0}^{k-2}2^{-nk}\left\|\frac{|h_j|}{\eta_{10}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}(k-j)\frac{\|\chi_{B_j}\|_{L^{p'(\cdot)}}}{\|\chi_k\|_{L^{p'(\cdot)}}}|B_k| \right)^{(q_2^2)_k} \\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(\sum_{j=0}^{k-2}(k-j)2^{(k-j)(\alpha_+-n\delta_{11})}\left\|\frac{2^{j\alpha_+}|h_j|}{\eta_{10}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{(q_2^2)_k},
\end{aligned}$$

Thus, noting that $h \in MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)$, $\lambda_1(q_1)_- = \lambda_2(q_2)_-$ and $\alpha_+ < n\delta_{11} + \lambda_1/(q_1)_+$, we obtain

$$\begin{aligned}
&2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\| \left(\frac{2^{k\alpha(\cdot)}|\sum_{j=0}^{k-2}[b,T](h_j)\chi_k|}{\|b\|_*\eta_{10}} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\{ \sum_{j=0}^{k-2}(k-j)2^{(k-j)(\alpha_+-n\delta_{11})}\left\|\left(\frac{2^{j\alpha_+}|h_j|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|^{\frac{1}{(q_1^2)_j}}_{L^{\frac{p(\cdot)}{q_1(\cdot)}}(\mathbb{R}^n)} \right\}^{(q_2^2)_k} \\
&\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\{ \sum_{j=0}^{k-2}(k-j)2^{(k-j)(\alpha_+-n\delta_{11})}\left(2^{j\lambda_1}2^{-j\lambda_1}\sum_{\ell=0}^{j}\left\|\left(\frac{2^{\ell\alpha_+}|h\chi_\ell|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}(\mathbb{R}^n)} \right)^{\frac{1}{(q_1^2)_j}} \right\}^{(q_2^2)} \\
&\leq C2^{(k-k_0)\lambda_2}\sum_{k=0}^{k_0}\left\{ \sum_{j=0}^{k-2}(k-j)2^{(k-j)(\alpha_+-n\delta_{11}-\lambda_1(q_1)_-)}\left(2^{-j\lambda_1}\sum_{\ell=0}^{j}\left\|\left(\frac{2^{\ell\alpha_+}|h\chi_\ell|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}(\mathbb{R}^n)} \right)^{\frac{1}{(q_1^2)_j}} \right\}^{(q_2^2)} \\
&\leq C\sum_{k=0}^{k_0}2^{(k-k_0)\lambda_2}\left(\sum_{j=0}^{k-2}(k-j)2^{(k-j)(\alpha_+-n\delta_{11}-\lambda_1/(q_1)_-)} \right)^{(q_2^2)_k} \leq C.
\end{aligned}$$

where

$$(q_1^2)_j = \begin{cases} (q_1)_-, & \left\| \frac{2^{j\alpha_+}|h\chi_j|}{\eta_{10}} \right\|_{L^{p(\cdot)}} \leq 1, \\ (q_1)_+, & \left\| \frac{2^{j\alpha_+}|h\chi_j|}{\eta_{10}} \right\|_{L^{p(\cdot)}} > 1. \end{cases}$$

This implies that

$$\eta_{22} \leq C\|b\|_* \eta_{10} \leq C\|b\|_* \|h\|_{MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)}.$$

Step 3. Finally, we η_{23}. Let $x \in R_k, y \in R_j$ and $j \geq k+2$. Since $\alpha_+ > -n\delta_{12} + \lambda_1/(q_1)_-$, by the similar argument in Step 2, we get

$$\begin{aligned}|[b,T]h_j(x)| &\leq \int_{R_j} |K(x,y)||b(x)-b(y)||h_j(y)|dy \leq C2^{-jn}\left[|b(x)-b_{B_j}|\int_{R_j}|h_j(y)|dy + \int_{R_j}|b(y)-b_{B_j}||h_j(y)|dy\right]\\ &\leq C2^{-jn}\left[|b(x)-b_{B_j}|\|h\|_{L^1(\mathbb{R}^n)} + \|(b-b_{B_j})h_j\|_{L^1(\mathbb{R}^n)}\right],\end{aligned}$$

and

$$\begin{aligned}&2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=k+2}^{\infty}[b,T](h_j)\chi_k|}{\|b\|_*\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}} \leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=k+2}^{\infty}[b,T](h_j)\chi_k|}{\|b\|_*\eta_{10}}\right\|_{L^{p(\cdot)}}^{(q_2^3)_k}\\ &\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=k+2}^{\infty}|b(x)-b_{B_j}|\|h\|_{L^1(\mathbb{R}^n)}\chi_k|}{\|b\|_*\eta_{10}}\right\|_{L^{p(\cdot)}}^{(q_2^3)_k} + C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=k+2}^{\infty}\|(b-b_{B_j})h_j\|_{L^1(\mathbb{R}^n)}\chi_k|}{\|b\|_*\eta_{10}}\right\|_{L^{p(\cdot)}}^{(q_2^3)_k}\\ &\leq C2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left(2^{k\alpha_+}\sum_{j=0}^{k-2}2^{-jn}\left\|\frac{|h_j|}{\eta_{10}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}(j-k)\frac{\|\chi_k\|_{L^{p(\cdot)}}}{\|\chi_{B_j}\|_{L^{p(\cdot)}}}|B_j|\right)^{(q_2^3)_k},\end{aligned}$$

where

$$(q_2^3)_k = \begin{cases}(q_2)_-, & \left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=k+2}^{\infty}[b,T](h_j)\chi_k|}{\eta_{10}\|b\|_*}\right\|_{L^{p(\cdot)}} \leq 1,\\ (q_2)_+, & \left\|\frac{2^{k\alpha(\cdot)}|\sum_{j=k+2}^{\infty}[b,T](h_j)\chi_k|}{\eta_{10}\|b\|_*}\right\|_{L^{p(\cdot)}} > 1.\end{cases}$$

Therefore

$$\begin{aligned}&2^{-k_0\lambda_2}\sum_{k=0}^{k_0}\left\|\left(\frac{2^{k\alpha(\cdot)}|\sum_{j=k+2}^{\infty}[b,T](h_j)\chi_k|}{\|b\|_*\eta_{10}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\\ &\leq C\sum_{k=0}^{k_0}2^{(k-k_0)\lambda_2}\left\{\sum_{j=k+2}^{\infty}(j-k)2^{(k-j)(\alpha_++n\delta_{12}-\lambda_1/(q_1)_-)}\left(2^{j\lambda_1}2^{-j\lambda_1}\sum_{\ell=0}^{j}\left\|\left(\frac{2^{\ell\alpha_+}|h\chi_\ell|}{\eta_{10}}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_1(\cdot)}}(\mathbb{R}^n)}\right)^{\frac{1}{(q_1^3)_j}}\right\}^{(q_2^3)_k}\\ &\leq C\sum_{k=0}^{k_0}2^{(k-k_0)\lambda_2}\left(\sum_{j=k+2}^{\infty}(j-k)2^{(k-j)(\alpha_++n\delta_{12}-\lambda_1/(q_1)_-)}\right)^{(q_2^3)_k}\\ &\leq C,\end{aligned}$$

which implies that

$$\eta_{23} \leq C\|b\|_* \eta_{10} \leq C\|b\|_* \|h\|_{MK^{\alpha_+,\lambda_1}_{q_1(\cdot),p(\cdot)}(\mathbb{R}^n)}.$$

Combining the above estimates for η_{21}, η_{22} and η_{23}, the get our desired result. □

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