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Boundary Value Technique for Initial Value Problems with Continuous Third Derivative Multistep Method of Enright

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Authors' contributions

This work was carried out in collaboration between both authors. Author IOL proposed the algotithms, author AOA developed, analysed and implemented the method. Authors IOL and AOA drafted the manuscripts. Both authors read and approved the final manuscript.

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Abstract

The Enright's third derivative method which is A-stable is derived using multistep collocation approach. The continuous method so obtained are use to generate the main method and the complementary methods to solve standard problems via boundary value techniques such that the numerical solution of a problem is obtained on the domain of integration simultaneously. Numerical result obtained via the implementation of the methods shows that the new method can compete with the existing ones (Enright [1], Ehigie, Jator, Sofolowe and Okunuga [2], Jator-Sahi [3] , Wu-Xia [4]) in the literature.

Keywords: Continuous schemes; multistep collocation; stiff system; initial value problem.

Mathematics Subject Classification (2000): Primary 65L05 Secondary 65L06.

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1 Introduction

Differential equations are important tools in solving real-world problems and a wide variety of natural phenomena are modelled into differential equations. Mathematical modelling of real-life problems usually result into functional equations, e.g Ordinary Differential Equation(ODE), Partial Differntial Equation, Integral and Integro Differential equation, Stochastic Differential Equation and others. ODEs which normally arises in biological models, circuit theory models, circuit theory models, fluid and chemical kinetics models may or may not have exact solutions, thus a need for a numerical solution.

Reactions in physical systems often transform into system of ODE, which some class of these system are called Stiff system. The numerical methods for obtaining solutions to class of problems are One step method and Multistep method(LMM).

Consider the model Stiff system ODE attached with initial condition of the form:

$$y' = Ay \qquad y(a) = y_0, \quad x \in [a, b] \tag{1.1}$$

where $y(x) \in \mathbb{R}^{\geqslant}$, A is an $m \times m$ real matrix with its eigenvalues $\{\lambda_j\}_{j=1}^{m}$, such that $\text{Re}(\lambda_j) < 0$ and the ratio

$$\frac{\max|Re(\lambda_j)|}{\min|Re(\lambda_j)|} >> 0$$

Several numerical methods such as the Finite Difference method(Brugnano and Trigiante [5]), Multistep collocation method (Adeniran, Odejide and Ogundare [6], Adeniran and Ogundare [7], Odejide and Adeniran [8]) have been proposed for numerical solution of equation (1.1). Multi-derivative method for solving systems of ODE was proposed by Obrenchkoff [9] and special cases of the Obrenchkoff were later proposed by Cash [10], Enright [1], Jia-Xiang, Jiao-Xun[11] and of recent Ehigie, Jator, Sofoluwe and Okunuga [2]. And of these methods, the justification for including higher term in such method was clearly stated by Enright [1], which will include method with higher order, to obtain stability at infinity and to obtain a method with reasonable stability properties on the neighborhood of the orgin. This class of Enright's schemes is a special class of the Obrenchkoff [9] methods which are found to be of order $p = k + 2$ for a k step method.

In this paper, a continuous form of the third derivative multistep method shall be derived through a multistep collocation technique. The method will be in block and will be implemented on IVP. This paper will further discuss the implementation of the newly developed methods using Boundary Value Methods as it was also implemented in Amodio and Mazzia [12], Axelsson and Verwer [13], Brugnano and Trigiante [5], Ehigie et al. [2], Jator and Sahi [3], so that the numerical solution $(y_1, y_2, \cdots, y_N)^T$ are obtained simultaneously. The advantage of this approach is that the global error at the end of the integration are smaller than the ones accumulated over various local truncation error obtained via step by step implementation; Lambert [14], more also the block method are known to relax the A-stability criteria of other higher methods; Axelsson and Verwer [13].

2 Theoretical Procedure

The third derivative multistep methods are derived using collocation technique as discussed in Adeniran, Ogundare and Odejide [6], Adeniran and Ogundare[7] and Ehigie et al. [2], Odejide and Adeniran [8].

Consider the intial value problem of the form

$$y' = f(x, y) \quad y(a) = y_0 \quad x \in [a, b] \tag{2.1}$$

The general third derivative formula for solving equation (2.1) using k-step third derivative linear multistep method is of the form.

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j} + h^2\sum_{j=0}^{k}\delta_j g_{n+j} + h^3\sum_{j=0}^{k}\chi_j e_{n+j} \tag{2.2}$$

$$\text{where} \quad y_{n+j} \approx y(x_n + jh),$$
$$f_{n+j} \equiv f(x_n + jh, y(x_n + jh))$$
$$g_{n+j} = \frac{df(x, y(x))}{dx}\Big|_{y=y_{n+j}}^{x=x_{n+j}},$$
$$e_{n+j} = \frac{dg(x, y(x))}{dx}\Big|_{y=y_{n+j}}^{x=x_{n+j}}$$

x_n is a discrete point at "n", α_j, β_j , γ_j and χ_j are coefficients to be determined. To obtained the method of the form (2.2), $y(x)$ is approximated by a basis polynomial of the form

$$y(x) = \sum_{j=0}^{m} a_j \left(\frac{x - x_n}{h}\right)^j \tag{2.3}$$

Equation (2.3) will be used for the derivation of the main and complementary methods for the two classes of continuous third derivatives multistep method of Enright which is a special case of (2.3) Interpolating $y(x)$ at point x_{n+k-1}, collocating $y'(x)$ at points x_{n+j}, $j = 0, 1, 2, \cdots, k$, collocating $y''(x)$ at point x_{n+k} and also collocating $y'''(x)$ at point x_{n+k}. i.e

$$y(x_{n+k-1}) = y_{n+k-1}$$
$$y'(x) = f_{n+j}$$
$$y''(x) = g_{n+j} \qquad j = 0, 1, 2, \cdots, k$$
$$y'''(x) = e_{n+j}$$

The system of equations generated are solved to obtained the coefficients of a_j, $j = 0, 1, 2, \cdots, k+2$ which are used to generate the continuous multistep method of Enright of the form

$$y(x) = y_{n+k-1} + h\sum_{j=0}^{k}\beta_j f_{n+j} + h^2\delta_k g_{n+k} + h^3\chi_k e_{n+k} \tag{2.4}$$

Evaluating (2.4) at $x = x_{n+k}$ yields the third derivative multistep method of Enright, evaluating at $x = x_{n+j}$, $j = 0, 1, 2, \cdots, k-2$ gives $(k-1)$ methods, which will be called complementary methods to complete the k block for the system. The Enright's method so obtained is of the form

$$y_{n+k} = y_{n+k-1} + h\sum_{j=0}^{k}\beta_j f_{n+j} + h^2\delta_k g_{n+k} + h^3\chi_k e_{n+k} \tag{2.5}$$

Thus, if (2.5) is associated with a linear difference operator,

$$\begin{aligned} L[y(x_n : h)] = {} & [y(x + kh) - y(x + (k-1)h - h\sum_{j=0}^{k}\beta_j y'(x_n + jh) - h^2\delta_k y''(x_n + kh) \\ & - h^3\chi_k y'''(x_n + kh)] \end{aligned} \tag{2.6}$$

where $y(x)$ is an arbitrary smooth function. $L[y(x_n : h)]$ is called Local Truncation Error(LTE) at x_{n+k} if y represent the solution of the IVP (2.1).

By Taylor's series expansion of $y(x+jh)$, $y'(x+jh)$, $y''(x+jh)$ and $y'''(x+jh)$, $j = 0, 1, 2, \cdot, k$, the $L[y(x_n : h)]$ is obtain in the form,

$$L[y(x_n : h)] = C_0 y(x) + C_1 h y'(x) + C_2 \frac{h^2}{2!} y''(x) + \cdots + C_q \frac{h^q}{q!} y^q(x) + \cdots$$

where

$$C_0 = \sum_{j=0}^{k} \alpha_j$$

$$C_1 = 1 - \sum_{j=0}^{k} \beta_j$$

$$C_2 = \frac{1}{2!}(-(k-1)^2 + k^2) - \sum_{j=0}^{k} j\beta_j - \delta_k$$

$$C_3 = \frac{1}{3!}(-(k-1)^3 + k^3) - \frac{1}{2!}\sum_{j=0}^{k} j^2\beta_j - k\delta_k - \chi_k$$

$$C_4 = \frac{1}{4!}(-(k-1)^4 + k^4) - \frac{1}{3!}\sum_{j=0}^{k} j^3\beta_j - \frac{1}{2!}k^2\delta_k - k\chi_k$$

$$\vdots \qquad \qquad \vdots \qquad \qquad \qquad \vdots$$

$$C_k = \frac{1}{q!}(-(k-1)^q + k^q) - \frac{1}{(q-1)!}\sum_{j=0}^{k} j^{(q-1)}\beta_j - \frac{1}{(q-2)!}k^{(q-2)}\delta_k - \frac{1}{(q-3)!}k^{q-3}\chi_k$$

Thus, method (2.5) is said to be of order 'p' if,

$$C_0 = C_1 = C_2 = C_3 = \cdots = C_p = 0, \quad C_{p+1} \neq 0 \tag{2.7}$$

C_{p+1} is the Error Constant and $C_{p+1}h^{p+1}y^{(p+1)}(x_n)$ is the LTE at point x_n. Hence, the LTE is given as

$$C_{p+1}h^{p+1}y^{(p+1)}(x_n) \tag{2.8}$$

2.1 Derivation of the continuous third derivative multistep method for $k = 2$

To derive the continuous third derivative multistep method of Enright, Let the basis function $y(x)$ be

$$y(x) = \sum_{j=0}^{5} a_j \left(\frac{(x - x_n)}{h}\right)^j \tag{2.9}$$

We interpolate (2.9) at point $x = x_{n+1}$, collocate $y'(x)$ at points x_n, x_{n+1}, x_{n+2}, collocate $y''(x)$ at point x_{n+2}, also collocate $y''(x)$ at point x_{n+2} , we obtain a system of equation represented in matrix form

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 4 & 12 & 32 & 80 \\ 0 & 0 & 2 & 12 & 48 & 160 \\ 0 & 0 & 0 & 6 & 48 & 240 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} y_{n+1} \\ hf_n \\ hf_{n+1} \\ hf_{n+2} \\ h^2 g_{n+2} \\ h^3 g_{n+2} \end{pmatrix} \tag{2.10}$$

Solving the system of equations (2.10) for a_i, $i = 0(1)5$, we obtained

$$a_0 = y_{n+1} - \frac{49}{160}hf_n - \frac{13}{10}hf_{n+1} + \frac{97}{160}hf_{n+2} - \frac{33}{80}h^2g_{n+2} + \frac{23}{240}h^3e_{n+2}$$
$$a_1 = hf_n$$
$$a_2 = -\frac{5}{4}hf_n + 4hf_{n+1} - \frac{11}{4}hf_{n+2} + 2h^2g_{n+2} - \frac{1}{2}h^3e_{n+2}$$
$$a_3 = \frac{3}{4}hf_n - 4hf_{n+1} + \frac{13}{4}hf_{n+2} - \frac{5}{2}h^2g_{n+2} + \frac{2}{3}h^3e_{n+2}$$
$$a_4 = -\frac{7}{32}hf_n + \frac{3}{2}hf_{n+1} - \frac{41}{32}hf_{n+2} + \frac{17}{16}h^2g_{n+2} - \frac{5}{16}h^3e_{n+2}$$
$$a_5 = \frac{1}{40}hf_n - \frac{1}{5}hf_{n+1} + \frac{7}{40}hf_{n+2} - \frac{3}{20}h^2g_{n+2} + \frac{1}{20}h^3e_{n+2}$$

Substituting the value of a_i's, $i = 0, 1, 2, \cdots, 5$ into (2.9), we obtianed the continuous third derivatives multistep method of the form:

$$\begin{aligned}
y(x) = y_{n+1} + h\left(-\frac{49}{160} + (\frac{x-x_n}{h}) - \frac{5}{4}(\frac{x-x_n}{h})^2 + \frac{3}{4}(\frac{x-x_n}{h})^3 - \frac{7}{32}(\frac{x-x_n}{h})^4 + \frac{1}{40}(\frac{x-x_n}{h})^5\right) f_n \\
+\left(-\frac{13}{10} + 4(\frac{x-x_n}{h})^2h - 4(\frac{x-x_n}{h})^3h + \frac{3}{2}(\frac{x-x_n}{h})^4h - \frac{1}{5}(\frac{x-x_n}{h})^5h\right) f_{n+1} + \\
h\left(+\frac{97}{160} - \frac{11}{4}(\frac{x-x_n}{h})^2 + \frac{13}{4}(\frac{x-x_n}{h})^3 - \frac{41}{32}(\frac{x-x_n}{h})^4 + \frac{7}{40}(\frac{x-x_n}{h})^5\right) f_{n+2} \\
+h^2\left(-\frac{33}{80}2(\frac{x-x_n}{h})^2 - \frac{5}{2}(\frac{x-x_n}{h})^3 + \frac{17}{16}(\frac{x-x_n}{h})^4 - \frac{3}{20}(\frac{x-x_n}{h})^5\right) g_{n+2} + \\
h^3\left(\frac{23}{240} - \frac{1}{2}(\frac{x-x_n}{h})^2 + \frac{2}{3}(\frac{x-x_n}{h})^3 - \frac{5}{16}(\frac{x-x_n}{h})^4 + \frac{1}{20}(\frac{x-x_n}{h})^5\right) e_{n+2}
\end{aligned} \tag{2.11}$$

Evaluating (2.11) at $x = x_{n+2}$, and $x = x_n$, we obtain the main method and the complimentary method.

$$y_{n+2} = y_{n+1} - \frac{h}{160}\left(f_n - 48f_{n+1} - 113f_{n+2}\right) - \frac{17}{80}h^2g_{n+2} + \frac{7}{240}h^3e_{n+2} \tag{2.12}$$

$$y_n = y_{n+1} - \frac{h}{160}\left(49f_n + 208f_{n+1} - 97f_{n+2}\right) - \frac{33}{80}h^2g_{n+2} + \frac{23}{240}h^3e_{n+2} \tag{2.13}$$

Equation (2.12) and (2.13) will be taken together as a Boundary Value Method for "k=2" and denoted by TCM2

2.2 Derivation of the continuous third derivative multistep method for $k = 3$

Following the same process as presented in the theoretical procedure and the derivation for $k = 2$, we obtain a continuous method for k=3 as follows

$$\begin{aligned}
y(x) = y_{n+1} + \\
h\left(\frac{251}{810} + (\frac{x-x_n}{h}) - \frac{5}{4}(\frac{x-x_n}{h})^2 + \frac{7}{9}(\frac{x-x_n}{h})^3 - \frac{7}{27}(\frac{x-x_n}{h})^4 + \frac{2}{45}(\frac{x-x_n}{h})^5 - \frac{1}{324}(\frac{x-x_n}{h})^6\right) f_n \\
+h\left(-\frac{553}{480} + \frac{27}{8}(\frac{x-x_n}{h})^2 - \frac{27}{8}(\frac{x-x_n}{h})^3 + \frac{45}{32}(\frac{x-x_n}{h})^4 - \frac{11}{40}(\frac{x-x_n}{h})^5 + \frac{1}{48}(\frac{x-x_n}{h})^6\right) f_{n+1} + \\
h\left(\frac{4}{3} - \frac{27}{4}(\frac{x-x_n}{h})^2 + 9(\frac{x-x_n}{h})^3 - \frac{9}{2}(\frac{x-x_n}{h})^4 + (\frac{x-x_n}{h})^5 - \frac{1}{12}(\frac{x-x_n}{h})^6\right) f_{n+2} +
\end{aligned}$$

$$h(-\frac{11293}{12960}+\frac{37}{8}(\frac{x-x_n}{h})^2h+\frac{37}{8}(\frac{x-x_n}{h})^2-\frac{461}{72}(\frac{x-x_n}{h})^3+\frac{2897}{864}(\frac{x-x_n}{h})^4-\frac{277}{360}(\frac{x-x_n}{h})^5+ \frac{85}{1296}(\frac{x-x_n}{h})^6)f_{n+3}+$$

$$h^2\left(\frac{259}{432}-\frac{13}{4}(\frac{x-x_n}{h})^2+\frac{55}{12}(\frac{x-x_n}{h})^3-\frac{355}{144}(\frac{x-x_n}{h})^4+\frac{7}{12}(\frac{x-x_n}{h})^5-\frac{11}{216}(\frac{x-x_n}{h})^6\right)g_{n+3}$$

$$+h^3\left(-\frac{97}{720}+\frac{3}{4}(\frac{x-x_n}{h})^2-\frac{13}{12}(\frac{x-x_n}{h})^3+\frac{29}{48}(\frac{x-x_n}{h})^4-\frac{3}{20}(\frac{x-x_n}{h})^5+\frac{1}{72}(\frac{x-x_n}{h})^6\right)e_{n+3} \tag{2.14}$$

Evaluating (2.14) at $x = \{x_{n+3}, x_n, x_{n+2}\}$,we generate the main method and two complementary method which complete the Boundary Value Method as

$$y_{n+3} = y_{n+1} - \frac{h}{810}(8f_n - 810f_{n+1} - 1080f_{n+2} - 251f_{n+3}) - \frac{1}{27}h^2g_{n+3} - \frac{1}{45}h^3e_{n+3} \tag{2.15}$$

$$y_{n+2} = y_{n+1} + h(-\frac{1}{90}f_n + \frac{61}{160}f_{n+1} + f_{n+2} + \frac{533}{1440}f_{n+3}) - \frac{11}{48}h^2g_{n+3} - \frac{11}{240}h^3e_{n+3} \tag{2.16}$$

$$y_n = y_{n+1} + \frac{h}{12960}(-400f_n + 27f_{n+1} + 17280f_{n+2} - 11293f_{n+3}) - \frac{259}{432}h^2g_{n+3} - \frac{97}{720}h^3e_{n+3} \tag{2.17}$$

Equations (2.15)-(2.17) will be called the Boundary Value Method for $k = 3$ denoted by TCM3

Table 1. Order and error constant of the methods

Method	Order	Error Constant (C_{p+1})
13	5	$-\frac{1}{1800}$
14	5	$-\frac{1}{200}$
16	6	$\frac{1}{1050}$
17	6	$\frac{59}{50400}$
18	6	$\frac{97}{16800}$

3 Properties of Enright's Methods

The order and error constant for the derived numerical method in this paper are presented in Table 1 above. The result follows from Equation (2.6), (2.7) and (2.8).

Definition 3.1 *The Region of Absolute Stability of the method given by* (2.12) *and* (2.15) *is the set of all points $z \in Z$ such that all roots of characteristic equation are of absolute value less than one.*

Applying method (2.12) and (2.15) to the test problem,

$$y' = \lambda y$$

substituting $z = \lambda h$, The characteristic equation for method (2.12) is obtained as

$$\left(1+\frac{113}{160}z+\frac{17}{80}z^2-\frac{7}{240}z^3\right)\xi^2-\left(1+\frac{48}{160}z\right)\xi+\frac{1}{160}z \tag{3.1}$$

while that of method (2.15) is

$$\left(1+\frac{251}{810}z+\frac{1}{27}z^2+\frac{1}{45}z^3\right)\xi^3-\left(\frac{1080}{810}z\right)\xi^2-(1+z)\xi+\frac{8}{810}z \tag{3.2}$$

The region of absolute stability of method (2.12) and (2.15) are given in Figs. 1 and 2 respectively.

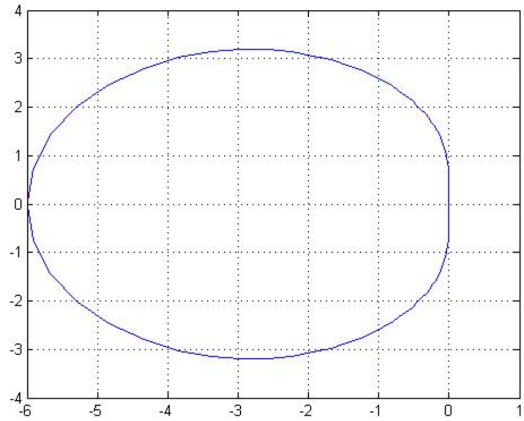

Fig. 1. Region of absolute stability for method (13)

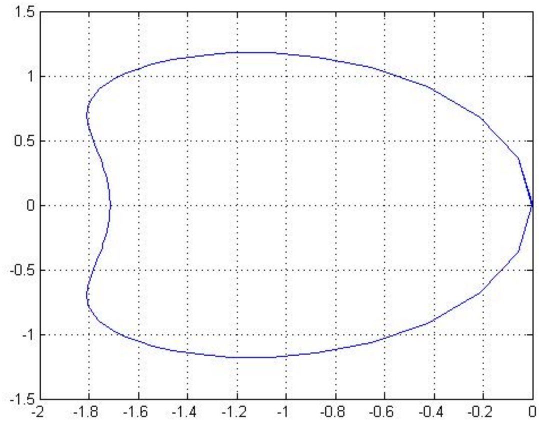

Fig. 2. Region of absolute stability for method(16)

4 Implementation Strategy of the Method

In the spirit of Baker and keech [15], a block by block method is a method for computing vectors $y_0, y_1, ...$ in sequence. The method (2.12, 2.13) and (2.15, 2.16, 2.17) are in block form and is applied in a block-by-block fashion. This is facilitated by the availability of the continuous representation, which generate a main discrete method and complimentary methods, which are combined and used as a block method to simultaneously produce approximations y_1,y_2, and y_1,y_2, y_3, for the solution of (2.1) at the points (x_1,x_2) and (x_1,x_2, x_3), in the first block. In order to apply the block method at the next block, the only necessary starting value is the last value of the previous block (y_2 and y_3 as the case may be) whose loss of accuracy do not affect subsequent points, thus the order of the algorithm is maintained. It is unnecessary to make a function evaluation at the initial part of the new block. Thus, at all blocks execept the first, the first function evaluation is already available from the previous block.

For linear problems, we solved equation (2.1) directly from the start with Gaussian elimination using partial pivoting and for non linear problems, we use a modified Newton-Raphson method.

5 Numerical Experiment

We solve the following examples to illustrate our new third derivative multistep method for various step sizes, All computations were carried out of the aid of MAPLE 13 software.

Experiment 5.1 The non-linear system solved by Wu and Xia [4] is considered:

$$\begin{aligned} y_1' &= -1002y_1 + 1000y_2^2 \qquad & y_1(0) = 1 \\ y_2' &= y_1 - 1000y_2(1 + y_2) \qquad & y_2(0) = 1 \end{aligned} \tag{5.1}$$

The exact solution of the system is given by:

$$y_1(x) = e^{-2x}, y_2(x) = e^{-x}.$$

The numerical result for Experiment 5.1 using step lenght (0.01) are presented in Tables 2, 3 and 4. The problem was compared to other existing methods. From Table 4, it is obvious that the Boundary Method for the third derivative method of Enright performed best as the computation proceed to x_N. where x_N are some point on the range of integration.

Table 2. Theoretical result for Experiment 5.1 with $h = 0.01$ (TCM2)

x	y_1	y_2
0.01	0.980198673890512	0.990049834317722
0.02	0.960789439738002	0.980198673875309
0.03	0.941764534169792	0.970445534116746
0.04	0.923116346954491	0.960789439676110
0.05	0.904837418604514	0.951229425024352

Table 3. Theoretical result for Experiment 5.1 with $h = 0.01$ (TCM3)

x	y_1	y_2
0.01	0.9801987321304870	0.990049839586492
0.02	0.960789497525562	0.980198679195128
0.03	0.941764539416637	0.970445539432241
0.04	0.923116352275009	0.960789444980707
0.05	0.904837556035959	0.951229430338037

Table 4. Numerical result for Experiment 5.1 (Maximum Error $Max|y_i - y(x_i))|$

Method	Number of steps	max $\|\|y_i - y(x_i)\|\|$	max $\|\|z_i - z(x_i)\|\|$
Wu-Xia [4]	500	2.56×10^{-07}	8.02×10^{-08}
Jator-Sahi[3]	125	1.63×10^{-14}	0.00
TCM2	500	4.22×10^{-13}	9.89×10^{-13}
TCM3	500	2.245×10^{-15}	5.00×10^{-14}

Experiment 5.2 We also consider the moderately stiff problem

$$\begin{aligned} y' &= -y - 10z, \qquad & y(0) = 1 \\ z' &= -10y - z \qquad & z(0) = 1 \end{aligned} \tag{5.2}$$

Exact solution $y(x) = e^{-x}\cos 10x, \;\; z(x) = e^{-x}\sin 10x$

The numerical result for Experiment 5.2 were presented in Tables 5 and 6. The problem was compared to other existing method. The new Boundary Method for the third derivative method of Enright displayed better accuracy within the range of integration.

Table 5. Numerical result for Experiment 5.2 (Maximum Error)

No. of steps	TCM2 $\max\|\|y_i - y(x_i)\|\|$	TCM2 $\max\|\|z_i - z(x_i)\|\|$	Ehigie et al. [2] $\max\|\|y_i - y(x_i)\|\|$	Ehigie et al. [2] $\max\|\|z_i - z(x_i)\|\|$
125	1.635×10^{-12}	4.795×10^{-12}	8.33×10^{-06}	1.32×10^{-6}
250	2.0×10^{-15}	1.6×10^{-14}	1.13×10^{-07}	1.36×10^{-08}
500	0	0	8.19×10^{-12}	6.30×10^{-12}

Table 6. Numerical result for Experiment 5.2 (Maximum Error)

No. of steps	TCM3 $\max\|\|y_i - y(x_i)\|\|$	TCM3 $\max\|\|z_i - z(x_i)\|\|$	Ehigie et al. [2] $\max\|\|y_i - y(x_i)\|\|$	Ehigie et al. [2] $\max\|\|z_i - z(x_i)\|\|$
125	8.56×10^{-13}	3.3×10^{-14}	8.33×10^{-06}	1.32×10^{-6}
250	1.0×10^{-15}	0	1.13×10^{-07}	1.36×10^{-08}
500	0	0	8.19×10^{-12}	6.30×10^{-12}

Experiment 5.3 The linear problem by Enright[9]

$$\begin{aligned} y_1 &= -0.1y_1 & y_1(0) &= 1 \\ y_2 &= -10y_2 & y_2(0) &= 1 \\ y_3 &= -100y_3 & y_3(0) &= 1 \\ y_4 &= -1000y_4 & y_4(0) &= 1 \end{aligned} \tag{5.3}$$

in the range $0 \le x \le 10$

Exact solution: $y_1 = e^{-0.10x}$, $y_2 = e^{-10x}$, $y_3 = e^{-100x}$, $y_4 = e^{-1000x}$

Table 7. Numerical result for Experiment 5.3 (Maximum Error)

x	step	y_1	y_2	y_3	y_4
Enright [1]	1000	4.1×10^{-7}	–	–	–
Ehigie et al. [2]	1000	4.2×10^{-16}	2.1×10^{-10}	0	0
TCM2	1000	0	0	0	0
TCM3	1000	0	0	0	0

The numerical result for Experiment 5.3 using step lenght (0.01) are presented in Table 7. it was observed that our derived method performed better and faster than that of Enright [1], Ehigie et al. [2], Jator and Sahi [3].

6 Conclusion

The method developed through collocation approach is capable of solving linear and non-linear first order ordinary differential equations and all other higher order differential equation when reduced to system of first order equations. The method is consistent and zero stable, satisfying the basic requirements for convergence of Linear Multistep methods (LMM). The methods displayed better accuracy when implemented with numerical examples than the method of Enright [1], Ehigie et al. [2], Jia-Xiang, Xiang and Jiao-xun Kuang [11].

We summarize the results as follows

Theorem 1: *The Enright's third derivative formula (TCM2 and TCM3) are A-stable and $O(h^6)$ and $O(h^7)$ respectively, efficient in term of numerical implementation and with a minimal error.*

Competing Interests

Authors have declared that no competing interests exist.

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