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Archimedean Operations-based Interval-valued Hesitant Fuzzy Ordered Weighted Operators and Their Application to Multi-criteria Decision Making

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

In this paper, the Archimedean t-conorm- and t-norm-based interval-valued hesitant fuzzy ordered weighted averaging (A-IVHFOWA) operator and the Archimedean t-conorm- and t-norm-based interval-valued hesitant fuzzy ordered weighted geometric (A-IVHFOWG) operator are given by taking fully account of the different weights associated with the particular ordered positions. Several desirable properties of the developed operators, such as commutativity, idempotency, and boundedness, are studied in detail, and some special cases of these operators are analyzed as well. Furthermore, we apply the proposed operators to develop a method for solving a multi-criteria decision making (MCDM) problem within the context of interval-valued hesitant fuzzy elements (IVHFEs). Finally, a practical example is provided to illustrate the practicality and effectiveness of the developed operators and method.

Keywords: Multi-criteria decision making; Hesitant fuzzy set; Interval-valued hesitant fuzzy set; t-conorm; t-norm; OWA; OWG.

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1 Introduction

In the multi-criteria decision making (MCDM) problems, the decision makers (DMs) often cannot provide their preference with single exact value, a margin of error or some possibility distribution on the possible values, but several possible values [1,2]. The fuzzy set [3] and it existing extensions, such as the intuitionistic fuzzy set [4], the interval-valued fuzzy set [5], the interval-valued intuitionistuc fuzzy set [6], the type-2 fuzzy set [7], and the fuzzy multiset [8,9], do not deal with the situation. To circumvent this issue, Torra [1] and Torra and Narukawa [2] proposed the concept of hesitant fuzzy sets (HFSs) to permits the membership degree of an element to a set to be presented as several possible values between 0 and 1. Since its first appearance, many scholars have paid great attentions to the HFSs [10-18]. For example, Xia and Xu [11] presented several operators for aggregating hesitant fuzzy information, and further investigated the correlations among these aggregation operators. Xia et al. [12] developed a series of quasi-arithmetic aggregation operators, ordered aggregation operator, induced aggregation operators for hesitant fuzzy information. Xu and Xia [13] proposed a variety of distance measures and ordered distance measures for hesitant fuzzy sets, and discussed their properties and relations as their parameters change. Xu and Xia [14] defined some distance measures and correlation coefficients for hesitant fuzzy elements, and investigated the differences and correlations among them in detail.

However, it is noted that the membership function in a HFS can only take the form of crisp numbers, and it is sometimes difficult to express the uncertain information. Later, Chen et al. [19,20] generalized the notion of HFSs by allowing the membership function to assume interval values, and defined the interval-valued hesitant fuzzy sets (IVHFSs). The core of an IVHFS is the interval-valued hesitant fuzzy number (IVHFN) [20], which is composed of some possible membership degree ranges. Interval-valued hesitant fuzzy numbers (IVHFNs) are a very useful tool to express a decision maker's preference information under uncertain or vague environments. With respect to MCDM problems in which criterion values take the form of IVHFEs, in order to get a decision result, an important step is the aggregation of IVHFNs. Until now, many different kinds of interval-valued hesitant fuzzy aggregation operators have been proposed to aggregate interval-valued hesitant fuzzy information. Chen et al. [20] developed a family of operators to fuse interval-valued hesitant fuzzy information, such as the IVHFWA operator, the IVHFWG operator, the GIVHFWA operator, the GIVHFWG operator, the IVHFOWA operator, the IVHFOWG operator, the GIVHFOWA operator, the GIVHFOWG operator, the IVHFHA operator, the IVHFHG operator, the GIVHFHA operator, and the GIVHFHG operator. Zhang et al. [21] developed several induced generalized interval-valued hesitant fuzzy operators, including the IGIVHFOWA operators and the IGIVHFOWG operator. It is clear that above aggregation operators are based on the algebraic operational laws of IVHFEs for carrying the combination process. The basic algebraic operations of IVHFEs are algebraic product and algebraic sum, which are not the only operations that can be chosen to model the intersection and union of IVHFEs. A generalized union and a generalized intersection on IVHFEs can be constructed from a general t-norm and t-conorm, i.e., the instances of various t-norms and t-conorms families can be used to perform the corresponding intersections and unions of IVHFEs. For an intersection, a good alternative to the algebraic product is the Einstein product, which typically gives the same smooth approximations as the algebraic product. Equivalently, for an intersection, a good alternative to the algebraic sum is the Einstein sum. Wei and Zhao [22] developed several new interval-valued hesitant fuzzy aggregation operators, such as the HIVFEWA operator, the HIVFEWG operator, the HIVFEOWA operator, the HIVFEOWG operator, the I-HIVFEOWA operator, and the I-HIVFEOWG operator.

The Archimedean t-conorm and t-norm [23,24] are generalizations of many other t-conorms and t-norms, such as the Algebraic, Einstein, Hamacher and Frank t-conorms and t-norms. The Archimedean t-conorm and t-norm are generated by an additive function $g(t)$ and its dual function $f(t)=g(1-t)$. When the additive generator $g(t)$ is assigned different forms, we can obtain some specific Archimedean t-conorms and t-norms. Thus, the Archimedean t-conorm and t-norm are more general and more flexible. Recently, Zhang and Wu [25] treated the interval-valued hesitant fuzzy aggregation operators with the help of Archimedean operations and developed two new Archimedean t-conorm- and t-norm-based interval-valued hesitant fuzzy aggregation operators, including the Archimedean t-conorm- and t-norm-based interval-valued hesitant fuzzy weighted averaging (A-IVHFWA) operator and the Archimedean t-conorm- and t-norm-based interval-valued hesitant fuzzy weighted geometric (A-IVHFWG) operator. Then, they applied these two operators to develop an approach for MCDM within the interval-valued hesitant fuzzy context and provided a practical example to demonstrate the proposed approach. It is noticed that the A-IVHFWA and A-IVHFWG operator weight only the interval-valued hesitant fuzzy arguments and take little account of the different weights of the particular ordered positions of arguments. This paper aims at introducing some "ordered" weighted operators to aggregate interval-valued hesitant fuzzy arguments based on the ideas of the OWA operator [26] and Archimedean operations, such as the Archimedean t-conorm- and t-norm-based interval-valued hesitant fuzzy ordered weighted averaging (A-IVHFOWA) operator and the Archimedean t-conorm- and t-norm-based interval-valued hesitant fuzzy ordered weighted geometric (A-IVHFOWG) operator. The prominent characteristic of the A-IVHFOWA and A-IVHFOWG operators is the reordering step in which the input arguments are rearranged in descending order, in particular, an interval-valued hesitant fuzzy argument is not associated with a particular weight but rather a weight is associated with a particular ordered position of the interval-valued hesitant fuzzy arguments, which are helpful for relieving the influence of unfair information on the decision result by assigning low weights to those ''false" or ''biased" arguments.

To do this, the rest of the paper is arranged as below. In Section 2, we review some basic knowledge. Section 3 develops two interval-valued hesitant fuzzy ordered weighted aggregation operators based on Archimedean t-conorm and t-norm. The properties and special cases of these newly developed aggregation operators are discussed as well. Section 4 gives a procedure to implement the proposed operators to MCDM within interval-valued hesitant fuzzy environments. Section 5 proposes a numerical example to verify the proposed method and compares it with the previous work. Concluding remarks and further research directions are included in Section 6.

2 Preliminaries

2.1 Hesitant fuzzy sets and interval-valued hesitant fuzzy sets

Definition 2.1 [1,2]. Let X be a reference set. A hesitant fuzzy set (HFS) A on X is in terms of a function $h_A(x)$ that when applied to X returns a subset of $[0,1]$ and is denoted by the following mathematical symbol [18]:

$$A=\left\{\left\langle x,h_A(x)\right\rangle \middle| x\in X\right\} \tag{1}$$

where $h_A(x)$ is a set of some values in $[0,1]$, which denote the possible membership degrees of the element $x\in X$ to the set A. For simiplity, Xia and Xu [11] called $h=h_A(x)$ a hesitant fuzzy element (HFE). Let H be the set of all HFEs.

Example 2.1. Let $X=\{x_1,x_2,x_3\}$, $A=\left\{\left\langle x_1,\{0.8,0.6\}\right\rangle,\left\langle x_2,\{0.4,0.3,0.2\}\right\rangle,\left\langle x_3,\{0.5,0.3\}\right\rangle\right\}$, and $h=\{0.4,0.3,0.2\}$. Then, A is a HFS on X and h is a HFE.

Definition 2.2 [20]. Let X be a fixed set and $D([0,1])$ be the set of all closed subintervals of $[0,1]$, i.e., $D([0,1])=\left\{a=\left[a^L,a^U\right] \middle| a^L\le a^U, a^L,a^U\in[0,1]\right\}$. An interval-valued hesitant fuzzy set (IVHFS) on X is in terms of a function that when applied to X returns a subset of $D([0,1])$.

Chen et al. [20] expressed the IVHFS as:

$$\tilde{A}=\left\{\left\langle x,\tilde{h}_{\tilde{A}}\left(x\right)\right\rangle \middle| x\in X\right\} \tag{2}$$

where $\tilde{h}_{\tilde{A}}\left(x\right)$ denotes all possible interval membership degrees of the element $x\in X$ to the set $\tilde{A}$. For simiplity, Chen et al. [20] called $\tilde{h}=\tilde{h}_{\tilde{A}}\left(x\right)$ an interval-valued hesitant fuzzy element (IVHFE). Let $\tilde{H}$ be the set of all interval-valued hesitant fuzzy elements (IVHFEs). If $\tilde{\gamma}\in\tilde{h}$, then $\tilde{\gamma}$ is an interval number [27] and can be denoted by $\tilde{\gamma}=\left[\tilde{\gamma}^L,\tilde{\gamma}^U\right]$, where $\tilde{\gamma}^L=\inf\tilde{\gamma}$ and $\tilde{\gamma}^U=\sup\tilde{\gamma}$ express the lower and upper limits of $\tilde{\gamma}$, respectively. Obviously, if $\tilde{\gamma}^L=\tilde{\gamma}^U$ for any $\tilde{\gamma}\in\tilde{h}$, then the IVHFEs are reduced to the HFEs.

Example 2.2. Let $X=\left\{x_1,x_2,x_3\right\}$,
$\tilde{A}=\left\{\left\langle x_1,\left\{[0.8,0.9],[0.5,0.6]\right\}\right\rangle,\left\langle x_2,\left\{[0.3,0.5],[0.3,0.4],[0.2,0.3]\right\}\right\rangle,\left\langle x_3,\left\{[0.6,0.8],[0.4,0.5]\right\}\right\rangle\right\}$,
and $\tilde{h}=\left\{[0.3,0.5],[0.3,0.4],[0.2,0.3]\right\}$. Then, $\tilde{A}$ is an IVHFS on X and $\tilde{h}$ is an IVHFE.

Zhang and Wu [25] put forward the following comparison laws for comparing the IVHFEs:

Definition 2.3 [25]. For an IVHFE $\tilde{h}=\left\{\tilde{\gamma}\middle|\tilde{\gamma}\in\tilde{h}\right\}=\left\{\left[\tilde{\gamma}^L,\tilde{\gamma}^U\right]\middle|\tilde{\gamma}\in\tilde{h}\right\}$, $s\left(\tilde{h}\right)=\dfrac{\sum_{\tilde{\gamma}\in\tilde{h}}\left(\tilde{\gamma}^L+\tilde{\gamma}^U\right)}{2l_{\tilde{h}}}$ is called the score function of $\tilde{h}$, where $l_{\tilde{h}}$ is the number of intervals in $\tilde{h}$.

Definition 2.4 [25]. For an IVHFE $\tilde{h}=\left\{\tilde{\gamma}\middle|\tilde{\gamma}\in\tilde{h}\right\}=\left\{\left[\tilde{\gamma}^L,\tilde{\gamma}^U\right]\middle|\tilde{\gamma}\in\tilde{h}\right\}$, $v\left(\tilde{h}\right)=\dfrac{\sum_{\tilde{\gamma}\in\tilde{h}}\left(\left|\tilde{\gamma}^L-s\left(\tilde{h}\right)\right|+\left|\tilde{\gamma}^U-s\left(\tilde{h}\right)\right|\right)}{2l_{\tilde{h}}}$ is referred to as the variance function of $\tilde{h}$, where $s\left(\tilde{h}\right)$ is the score function of $\tilde{h}$.

Definition 2.5 [25]. Let $\tilde{h}_1$ and $\tilde{h}_2$ be any two IVHFEs, and let $s\left(\tilde{h}_i\right)$ and $v\left(\tilde{h}_i\right)$ ($i=1,2$) be the score functions and the variance functions of $\tilde{h}_i$ ($i=1,2$), respectively. Then, the following conditions hold:

(1) If $s\left(\tilde{h}_1\right)>s\left(\tilde{h}_2\right)$, then $\tilde{h}_1>\tilde{h}_2$.
(2) If $s\left(\tilde{h}_1\right)=s\left(\tilde{h}_2\right)$, then

① if $v\left(\tilde{h}_1\right)<v\left(\tilde{h}_2\right)$, then $\tilde{h}_1>\tilde{h}_2$.
② if $v\left(\tilde{h}_1\right)=v\left(\tilde{h}_2\right)$, then $\tilde{h}_1=\tilde{h}_2$.

For three IVHFEs $\tilde{h}$, $\tilde{h}_1$, and $\tilde{h}_2$, Chen et al. [20] developed several operational laws for them as:

(1) $\tilde{h}^c = \left\{ \left[1-\tilde{\gamma}^U, 1-\tilde{\gamma}^L \right] \middle| \tilde{\gamma} \in \tilde{h} \right\}$

(2) $\tilde{h}_1 \cup \tilde{h}_2 = \left\{ \left[\tilde{\gamma}_1^L \vee \tilde{\gamma}_2^L, \tilde{\gamma}_1^U \vee \tilde{\gamma}_2^U \right] \middle| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2 \right\}$

(3) $\tilde{h}_1 \cap \tilde{h}_2 = \left\{ \left[\tilde{\gamma}_1^L \wedge \tilde{\gamma}_2^L, \tilde{\gamma}_1^U \wedge \tilde{\gamma}_2^U \right] \middle| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2 \right\}$

2.2 Archimedean t-norm and Archimedean t-conorm

Definition 2.6 [23,24]. A function $T:[0,1]\times[0,1]\to[0,1]$ is called a t-norm if it satisfies the following four conditions:

(1) $T(1,a)=a$ for all $a\in[0,1]$.

(2) $T(a,b)=T(b,a)$ for all $a,b\in[0,1]$.

(3) $T(a,T(b,c))=T(T(a,b),c)$ for all $a,b,c\in[0,1]$.

(4) If $a\le a'$ and $b\le b'$ for all $a,a',b,b'\in[0,1]$, then $T(a,b)\le T(a',b')$.

Definition 2.7 [23,24]. A function $S:[0,1]\times[0,1]\to[0,1]$ is called a t-conorm if it satisfies the following four conditions:

(1) $S(0,a)=a$ for all $a\in[0,1]$.

(2) $S(a,b)=S(b,a)$ for all $a,b\in[0,1]$.

(3) $S(a,S(b,c))=S(S(a,b),c)$ for all $a,b,c\in[0,1]$.

(4) If $a\le a'$ and $b\le b'$ for all $a,a',b,b'\in[0,1]$, then $S(a,b)\le S(a',b')$.

Definition 2.8 [23,24]. A t-norm function $T(a,b)$ is called an Archimedean t-norm if it is continuous and $T(a,a)<a$ for all $a\in(0,1)$. An Archimedean t-norm is called a strictly Archimedean t-norm if it is strictly increasing in each variable for $a,b\in(0,1)$.

Definition 2.9 [23,24]. A t-conorm function $S(a,b)$ is called an Archimedean t-conorm if it is continuous and $S(a,a)>a$ for all $a\in(0,1)$. An Archimedean t-conorm is called a strictly Archimedean t-conorm if it is strictly increasing in each variable for $a,b\in(0,1)$.

It is well known [28] that a strict Archimedean t-norm $T(a,b)$ is characterized by its additive generator g as $T(a,b)=g^{-1}(g(a)+g(b))$, where $g:[0,1]\to[0,+\infty]$ is a strictly decreasing function such that $g(1)=0$. A dual Archimedean t-conorm $S(a,b)$ is expressed as $S(a,b)=f^{-1}(f(a)+f(b))$ with $f(t)=g(1-t)$. Clearly, we have $f^{-1}(t)=1-g^{-1}(t)$.

2.3 Archimedean t-norm- and Archimedean t-conorm-based operational laws for IVHFEs

To aggregate the interval-valued hesitant fuzzy information, Zhang and Wu [25] used the Archimedean t-norm and Archimedean t-conorm to proposed several new operational laws for IVHFEs.

Definition 2.10. Given three IVHFEs $\tilde{h}$, $\tilde{h}_1$ and $\tilde{h}_2$, we define the following operational laws:

(1) $$\tilde{h}_1 \oplus \tilde{h}_2 = \left\{ \left[S\left(\tilde{\gamma}_1^L, \tilde{\gamma}_2^L\right), S\left(\tilde{\gamma}_1^U, \tilde{\gamma}_2^U\right) \right] \middle| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2 \right\}$$
$$= \left\{ \left[f^{-1}\left(f\left(\tilde{\gamma}_1^L\right) + f\left(\tilde{\gamma}_2^L\right)\right), f^{-1}\left(f\left(\tilde{\gamma}_1^U\right) + f\left(\tilde{\gamma}_2^U\right)\right) \right] \middle| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2 \right\}$$

(2) $$\tilde{h}_1 \otimes \tilde{h}_2 = \left\{ \left[T\left(\tilde{\gamma}_1^L, \tilde{\gamma}_2^L\right), T\left(\tilde{\gamma}_1^U, \tilde{\gamma}_2^U\right) \right] \middle| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2 \right\}$$
$$= \left\{ \left[g^{-1}\left(g\left(\tilde{\gamma}_1^L\right) + g\left(\tilde{\gamma}_2^L\right)\right), g^{-1}\left(g\left(\tilde{\gamma}_1^U\right) + g\left(\tilde{\gamma}_2^U\right)\right) \right] \middle| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2 \right\}$$

(3) $$\lambda \tilde{h} = \left\{ \left[f^{-1}\left(\lambda f\left(\tilde{\gamma}^L\right)\right), f^{-1}\left(\lambda f\left(\tilde{\gamma}^U\right)\right) \right] \middle| \tilde{\gamma} \in \tilde{h} \right\},\ \lambda > 0$$

(4) $$\tilde{h}^{\lambda} = \left\{ \left[g^{-1}\left(\lambda g\left(\tilde{\gamma}^L\right)\right), g^{-1}\left(\lambda g\left(\tilde{\gamma}^U\right)\right) \right] \middle| \tilde{\gamma} \in \tilde{h} \right\}\ \lambda > 0$$

3 Interval-valued Hesitant Fuzzy Ordered Aggregation Operators

It is well known that the ordered weighted averaging (OWA) operator first introduced by Yager [26] has achieved successful applications in many domains [29-35]. In this section, by combining the ordered weighted averaging (OWA) operator [26] with the operational laws given in Definition 2.10, some new operators to aggregate IVHFEs are developed, whose fundamental characteristics are also the reordering steps, and then their desirable properties are discussed.

Definition 3.1. Let $\tilde{h}_i$ ($i = 1, 2, \cdots, n$) be a collection of IVHFEs, $\tilde{h}_{\sigma(i)}$ be the ith largest of them, and let $w = \left(w_1, w_2, \cdots, w_n\right)^T$ be the aggregation-associated vector satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. An Archimedean t-conorm- and t-norm-based interval-valued hesitant fuzzy ordered weighted averaging (A-IVHFOWA) operator is a mapping $\tilde{H}^n \to \tilde{H}$ such that

$$\text{A-IVHFOWA}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right) = \mathop{\oplus}_{i=1}^{n} \left(w_i \tilde{h}_{\sigma(i)}\right) \tag{3}$$

Theorem 3.1. Let $\tilde{h}_i$ ($i = 1, 2, \cdots, n$) be a collection of IVHFEs, $\tilde{h}_{\sigma(i)}$ be the ith largest of them, and let $w = \left(w_1, w_2, \cdots, w_n\right)^T$ be the aggregation-associated vector satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$; then, the aggregated value by using the A-IVHFOWA operator is also an IVHFE, and

$$\text{A-IVHFOWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)$$
$$=\left\{\left[f^{-1}\left(\sum_{i=1}^{n}w_i f\left(\tilde{\gamma}_{\sigma(i)}^{L}\right)\right), f^{-1}\left(\sum_{i=1}^{n}w_i f\left(\tilde{\gamma}_{\sigma(i)}^{U}\right)\right)\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\tilde{\gamma}_{\sigma(2)}\in\tilde{h}_{\sigma(2)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\} \quad (4)$$

Proof. For $n=2$, because

$$w_1\tilde{h}_{\sigma(1)}=\left\{\left[f^{-1}\left(w_1 f\left(\tilde{\gamma}_{\sigma(1)}^{L}\right)\right), f^{-1}\left(w_1 f\left(\tilde{\gamma}_{\sigma(1)}^{U}\right)\right)\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)}\right\}$$
$$w_2\tilde{h}_{\sigma(2)}=\left\{\left[f^{-1}\left(w_2 f\left(\tilde{\gamma}_{\sigma(2)}^{L}\right)\right), f^{-1}\left(w_2 f\left(\tilde{\gamma}_{\sigma(2)}^{U}\right)\right)\right]\middle|\tilde{\gamma}_{\sigma(2)}\in\tilde{h}_{\sigma(2)}\right\}$$

we have

$$w_1\tilde{h}_1\oplus w_2\tilde{h}_2$$
$$=\left\{\left[f^{-1}\left(w_1 f\left(\tilde{\gamma}_{\sigma(1)}^{L}\right)\right), f^{-1}\left(w_1 f\left(\tilde{\gamma}_{\sigma(1)}^{U}\right)\right)\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)}\right\}\oplus\left\{\left[f^{-1}\left(w_2 f\left(\tilde{\gamma}_{\sigma(2)}^{L}\right)\right), f^{-1}\left(w_2 f\left(\tilde{\gamma}_{\sigma(2)}^{U}\right)\right)\right]\middle|\tilde{\gamma}_{\sigma(2)}\in\tilde{h}_{\sigma(2)}\right\}$$
$$=\left\{\left[\begin{array}{l}f^{-1}\left(f\left(f^{-1}\left(w_1 f\left(\tilde{\gamma}_{\sigma(1)}^{L}\right)\right)\right)+f\left(f^{-1}\left(w_2 f\left(\tilde{\gamma}_{\sigma(2)}^{L}\right)\right)\right)\right),\\ f^{-1}\left(f\left(f^{-1}\left(w_1 f\left(\tilde{\gamma}_{\sigma(1)}^{U}\right)\right)\right)+f\left(f^{-1}\left(w_2 f\left(\tilde{\gamma}_{\sigma(2)}^{U}\right)\right)\right)\right)\end{array}\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\tilde{\gamma}_{\sigma(2)}\in\tilde{h}_{\sigma(2)}\right\}$$
$$=\left\{\left[f^{-1}\left(w_1 f\left(\tilde{\gamma}_{\sigma(1)}^{L}\right)+w_2 f\left(\tilde{\gamma}_{\sigma(2)}^{L}\right)\right), f^{-1}\left(w_1 f\left(\tilde{\gamma}_{\sigma(1)}^{U}\right)+w_2 f\left(\tilde{\gamma}_{\sigma(2)}^{U}\right)\right)\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\tilde{\gamma}_{\sigma(2)}\in\tilde{h}_{\sigma(2)}\right\}$$

That is, the Eq. (4) holds for $n=2$. Suppose that the Eq. (4) holds for $n=k$, i.e.,

$$\mathop{\oplus}_{i=1}^{k}\left(w_i\tilde{h}_i\right)=\left\{\left[f^{-1}\left(\sum_{i=1}^{k}w_i f\left(\tilde{\gamma}_{\sigma(i)}^{L}\right)\right), f^{-1}\left(\sum_{i=1}^{k}w_i f\left(\tilde{\gamma}_{\sigma(i)}^{U}\right)\right)\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\tilde{\gamma}_{\sigma(2)}\in\tilde{h}_{\sigma(2)},\cdots,\tilde{\gamma}_{\sigma(k)}\in\tilde{h}_{\sigma(k)}\right\}$$

then, when $n=k+1$, we have

$$\mathop{\oplus}_{i=1}^{k+1}\left(w_i\tilde{h}_{\sigma(i)}\right)=\left(\mathop{\oplus}_{i=1}^{k}\left(w_i\tilde{h}_{\sigma(i)}\right)\right)\oplus\left(w_{k+1}\tilde{h}_{\sigma(k+1)}\right)$$
$$=\left\{\left[f^{-1}\left(\sum_{i=1}^{k}w_i f\left(\tilde{\gamma}_{\sigma(i)}^{L}\right)\right), f^{-1}\left(\sum_{i=1}^{k}w_i f\left(\tilde{\gamma}_{\sigma(i)}^{U}\right)\right)\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\tilde{\gamma}_{\sigma(2)}\in\tilde{h}_{\sigma(2)},\cdots,\tilde{\gamma}_{\sigma(k)}\in\tilde{h}_{\sigma(k)}\right\}\oplus$$
$$\left\{\left[f^{-1}\left(w_{k+1} f\left(\tilde{\gamma}_{\sigma(k+1)}^{L}\right)\right), f^{-1}\left(w_{k+1} f\left(\tilde{\gamma}_{\sigma(k+1)}^{U}\right)\right)\right]\middle|\tilde{\gamma}_{\sigma(k+1)}\in\tilde{h}_{\sigma(k+1)}\right\}$$
$$=\left\{\left[\begin{array}{l}f^{-1}\left(f\left(f^{-1}\left(\sum_{i=1}^{k}w_i f\left(\tilde{\gamma}_{\sigma(i)}^{L}\right)\right)\right)+f\left(f^{-1}\left(w_{k+1} f\left(\tilde{\gamma}_{\sigma(k+1)}^{L}\right)\right)\right)\right),\\ f^{-1}\left(f\left(f^{-1}\left(\sum_{i=1}^{k}w_i f\left(\tilde{\gamma}_{\sigma(i)}^{U}\right)\right)\right)+f\left(f^{-1}\left(w_{k+1} f\left(\tilde{\gamma}_{\sigma(k+1)}^{U}\right)\right)\right)\right)\end{array}\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\tilde{\gamma}_{\sigma(2)}\in\tilde{h}_{\sigma(2)},\cdots,\tilde{\gamma}_{\sigma(k)}\in\tilde{h}_{\sigma(k)},\tilde{\gamma}_{\sigma(k+1)}\in\tilde{h}_{\sigma(k+1)}\right\}$$
$$=\left\{\left[f^{-1}\left(\sum_{i=1}^{k+1}w_i f\left(\tilde{\gamma}_{\sigma(i)}^{L}\right)\right), f^{-1}\left(\sum_{i=1}^{k+1}w_i f\left(\tilde{\gamma}_{\sigma(i)}^{U}\right)\right)\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\tilde{\gamma}_{\sigma(2)}\in\tilde{h}_{\sigma(2)},\cdots,\tilde{\gamma}_{\sigma(k)}\in\tilde{h}_{\sigma(k)},\tilde{\gamma}_{\sigma(k+1)}\in\tilde{h}_{\sigma(k+1)}\right\}$$

i.e., Eq. (4) holds for $n=k+1$. Thus, Eq. (4) holds for all n.

In addition, since $f:[0,1]\rightarrow[0,+\infty]$ is a strictly increasing function, $f^{-1}:[0,+\infty]\rightarrow[0,1]$ exists and is also a strictly increasing function. Thus, by Eq. (4), for any $\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\tilde{\gamma}_{\sigma(2)}\in\tilde{h}_{\sigma(2)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}$, we have

$$0=f^{-1}\left(\sum_{i=1}^{n}w_i f(0)\right)\leq f^{-1}\left(\sum_{i=1}^{n}w_i f\left(\tilde{\gamma}_{\sigma(i)}^{L}\right)\right)\leq f^{-1}\left(\sum_{i=1}^{n}w_i f\left(\tilde{\gamma}_{\sigma(i)}^{U}\right)\right)\leq f^{-1}\left(\sum_{i=1}^{n}w_i f(1)\right)=1$$

which implies that $\text{A-IVHFOWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)$ is an IVHFE. The proof of Theorem 3.1 is completed. □

Especially, when $w=\left(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\right)^T$, the A-IVHFOWA operator is reduced to the A-IVHFA operator [25]:

$$\begin{aligned}&\text{A-IVHFA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\bigoplus_{i=1}^{n}\left(\frac{1}{n}\tilde{h}_i\right)\\&=\left\{\left[f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}f\left(\tilde{\gamma}_i^{L}\right)\right),f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}f\left(\tilde{\gamma}_i^{U}\right)\right)\right]\middle|\tilde{\gamma}_1\in\tilde{h}_1,\tilde{\gamma}_2\in\tilde{h}_2,\cdots,\tilde{\gamma}_n\in\tilde{h}_n\right\}\end{aligned}\tag{5}$$

The A-IVHFWA operator [25] weights only the interval-valued hesitant fuzzy arguments. However, by Definition 3.1, the A-IVHFOWA operator weights the ordered positions of the interval-valued hesitant fuzzy arguments instead of weighting the interval-valued hesitant fuzzy arguments themselves. The prominent characteristic of the A-IVHFOWA operator is the reordering step in which the input arguments are rearranged in descending order, in particular, an interval-valued hesitant fuzzy argument $\tilde{h}_i$ is not associated with a particular weight w_i but rather a weight w_i is associated with a particular ordered position i of the interval-valued hesitant fuzzy arguments.

Example 3.1. Assume that $\tilde{h}_1=\{[0.7,0.8],[0.5,0.6],[0.3,0.4]\}$, $\tilde{h}_2=\{[0.7,0.9],[0.3,0.5]\}$, and $\tilde{h}_3=\{[0.6,0.8],[0.2,0.3]\}$ are three IVHFEs, and the aggregation-associated vector is $w=(0.2,0.5,0.3)^T$. From Definition 2.3, we can calculate the score values of $\tilde{h}_1$, $\tilde{h}_2$, and $\tilde{h}_3$ as follows:

$$s\left(\tilde{h}_1\right)=\frac{(0.7+0.8)+(0.5+0.6)+(0.3+0.4)}{2\times 3}=0.55,\quad s\left(\tilde{h}_2\right)=\frac{(0.7+0.9)+(0.3+0.5)}{2\times 2}=0.6,$$

$$s\left(\tilde{h}_3\right)=\frac{(0.6+0.8)+(0.2+0.3)}{2\times 2}=0.475.$$

Since $s\left(\tilde{h}_2\right)>s\left(\tilde{h}_1\right)>s\left(\tilde{h}_3\right)$, then $\tilde{h}_{\sigma(1)}=\tilde{h}_2=\{[0.7,0.9],[0.3,0.5]\}$, $\tilde{h}_{\sigma(2)}=\tilde{h}_1=\{[0.7,0.8],[0.5,0.6],[0.3,0.4]\}$, and $\tilde{h}_{\sigma(3)}=\tilde{h}_3=\{[0.6,0.8],[0.2,0.3]\}$.

Suppose that $g(t)=\exp\left(\frac{1-t}{t}\right)-1$, then $f(t)=\exp\left(\frac{t}{1-t}\right)-1$, $g^{-1}(t)=\frac{1}{1+\log(1+t)}$, $f^{-1}(t)=\frac{\log(1+t)}{1+\log(1+t)}$, and the t-conorm and t-norm generated by $g(t)$ and $f(t)$ are as follows:

$$S(a,b)=\frac{\log\left(e^{\frac{a}{1-a}}+e^{\frac{b}{1-b}}-1\right)}{1+\log\left(e^{\frac{a}{1-a}}+e^{\frac{b}{1-b}}-1\right)},\qquad T(a,b)=\frac{1}{1+\log\left(e^{\frac{1-a}{a}}+e^{\frac{1-b}{b}}-1\right)}$$

Then, according to Eq. (4), we can obtain

$$\begin{aligned}&\text{A-IVHFOWA}\left(\tilde{h}_1,\tilde{h}_2,\tilde{h}_3\right)\\&=\left\{\begin{matrix}[0.6823,0.8812],[0.6698,0.8811],[0.6096,0.8810],[0.5721,0.8808],[0.5883,0.8810],[0.5387,0.8808],\\ [0.6573,0.7912],[0.6385,0.7697],[0.5243,0.7470],[0.4181,0.5407],[0.4690,0.7426],[0.2745,0.4055]\end{matrix}\right\}\end{aligned}$$

Several properties of the A-IVHFOWA operator are presented as follows:

Theorem 3.2. Let $\tilde{h}_i$ ($i=1,2,\cdots,n$) be a collection of IVHFEs. If $\tilde{h}'_1,\tilde{h}'_2,\cdots,\tilde{h}'_n$ is any permutation of $\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n$, then we have

$$\text{A-IVHFOWA}\left(\tilde{h}'_1,\tilde{h}'_2,\cdots,\tilde{h}'_n\right)=\text{A-IVHFOWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right) \tag{6}$$

which is called the commutativity.

Proof. If $\tilde{h}_{\sigma(i)}$ is the ith largest of $\tilde{h}_i$ ($i=1,2,\cdots,n$) and $\tilde{h}'_{\sigma(i)}$ is the ith largest of $\tilde{h}'_i$ ($i=1,2,\cdots,n$), respectively, then $\tilde{h}_{\sigma(i)}=\tilde{h}'_{\sigma(i)}$. Hence, $\text{A-IVHFOWA}\left(\tilde{h}'_1,\tilde{h}'_2,\cdots,\tilde{h}'_n\right)=\text{A-IVHFOWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)$. This completes the proof. □

Theorem 3.2 shows that the A-IVHFOWA operator is robust to permutations of the input IVHFEs and is independent of the input IVHFE labels.

Theorem 3.3. Let $\tilde{h}_i$ ($i=1,2,\cdots,n$) be a collection of IVHFEs. If all $\tilde{h}_i$ ($i=1,2,\cdots,n$) are equal, i.e., $\tilde{h}_i=\tilde{h}=\left\{\tilde{\gamma}\middle|\tilde{\gamma}\in\tilde{h}\right\}=\left\{\left[\tilde{\gamma}^L,\tilde{\gamma}^U\right]\middle|\tilde{\gamma}\in\tilde{h}\right\}$, for all i, then

$$\text{A-IVHFOWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\text{A-IVHFOWA}\left(\tilde{h},\tilde{h},\cdots,\tilde{h}\right)=\tilde{h} \tag{7}$$

which is called the idempotency.

Proof. According to Definition 3.1, we have

$$\text{A-IVHFOWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\overset{n}{\underset{i=1}{\oplus}}\left(w_i\tilde{h}_{\sigma(i)}\right)=\overset{n}{\underset{i=1}{\oplus}}\left(w_i\tilde{h}\right)=\left(\sum_{i=1}^{n}w_i\right)\tilde{h}=\tilde{h}$$

Thus, $\text{A-IVHFOWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\tilde{h}$. □

Theorem 3.4. For a collection of IVHFEs $\tilde{h}_i=\left\{\tilde{\gamma}_i\,\middle|\,\tilde{\gamma}_i\in\tilde{h}_i\right\}=\left\{\left[\tilde{\gamma}_i^L,\tilde{\gamma}_i^U\right]\,\middle\|\,\tilde{\gamma}_i\in\tilde{h}_i\right\}$ ($i=1,2,\cdots,n$), let $\tilde{h}^-=\left\{\left[\min_{i=1}^{n}\min_{\tilde{\gamma}_i\in\tilde{h}_i}\tilde{\gamma}_i^L,\min_{i=1}^{n}\min_{\tilde{\gamma}_i\in\tilde{h}_i}\tilde{\gamma}_i^U\right]\right\}$, $\tilde{h}^+=\left\{\left[\max_{i=1}^{n}\max_{\tilde{\gamma}_i\in\tilde{h}_i}\tilde{\gamma}_i^L,\max_{i=1}^{n}\max_{\tilde{\gamma}_i\in\tilde{h}_i}\tilde{\gamma}_i^U\right]\right\}$; then,

$$\tilde{h}^-\le\text{A-IVHFOWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)\le\tilde{h}^+ \tag{8}$$

which is called the boundedness.

Proof. For the simplicity of presentation, let $\tilde{\gamma}_-^L=\min_{i=1}^{n}\min_{\tilde{\gamma}_i\in\tilde{h}_i}\tilde{\gamma}_i^L$, $\tilde{\gamma}_-^U=\min_{i=1}^{n}\min_{\tilde{\gamma}_i\in\tilde{h}_i}\tilde{\gamma}_i^U$, $\tilde{\gamma}_+^L=\max_{i=1}^{n}\max_{\tilde{\gamma}_i\in\tilde{h}_i}\tilde{\gamma}_i^L$, $\tilde{\gamma}_+^U=\max_{i=1}^{n}\max_{\tilde{\gamma}_i\in\tilde{h}_i}\tilde{\gamma}_i^U$, $\tilde{h}=\text{A-IVHFOWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)$, $\tilde{\gamma}^L=f^{-1}\left(\sum_{i=1}^{n}w_i f\left(\tilde{\gamma}_{\sigma(i)}^L\right)\right)$, and $\tilde{\gamma}^U=f^{-1}\left(\sum_{i=1}^{n}w_i f\left(\tilde{\gamma}_{\sigma(i)}^U\right)\right)$. Then, $\tilde{h}^-=\left\{\left[\tilde{\gamma}_-^L,\tilde{\gamma}_-^U\right]\right\}$, $\tilde{h}^+=\left\{\left[\tilde{\gamma}_+^L,\tilde{\gamma}_+^U\right]\right\}$, and $\text{A-IVHFOWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\tilde{h}=\left\{\left[\tilde{\gamma}^L,\tilde{\gamma}^U\right]\,\middle\|\,\tilde{\gamma}\in\tilde{h}\right\}$.

For any $i=1,2,\cdots,n$, we have $\tilde{\gamma}_-^L=\min_{i=1}^{n}\min_{\tilde{\gamma}_i\in\tilde{h}_i}\tilde{\gamma}_i^L\le\tilde{\gamma}_{\sigma(i)}^L\le\max_{i=1}^{n}\max_{\tilde{\gamma}_i\in\tilde{h}_i}\tilde{\gamma}_i^L=\tilde{\gamma}_+^L$. Since $f(t)$, ($t\in[0,1]$) is a monotonic increasing function, we get

$$f^{-1}\left(\sum_{i=1}^{n}w_i f\left(\tilde{\gamma}_-^L\right)\right)\le f^{-1}\left(\sum_{i=1}^{n}w_i f\left(\tilde{\gamma}_{\sigma(i)}^L\right)\right)\le f^{-1}\left(\sum_{i=1}^{n}w_i f\left(\tilde{\gamma}_+^L\right)\right)$$

which is equivalent to

$$\tilde{\gamma}_-^L\le\tilde{\gamma}^L\le\tilde{\gamma}_+^L \tag{9}$$

Similarly, we have

$$\tilde{\gamma}_-^U\le\tilde{\gamma}^U\le\tilde{\gamma}_+^U \tag{10}$$

According to Definition 2.3, $s\left(\tilde{h}\right)=\frac{\sum_{\tilde{\gamma}\in\tilde{h}}\left(\tilde{\gamma}^L+\tilde{\gamma}^U\right)}{2l_{\tilde{h}}}$, $s\left(\tilde{h}^-\right)=\frac{\tilde{\gamma}_-^L+\tilde{\gamma}_-^U}{2}$, and $s\left(\tilde{h}^+\right)=\frac{\tilde{\gamma}_+^L+\tilde{\gamma}_+^U}{2}$. From Eqs. (9) and (10), we have

$$\frac{\tilde{\gamma}_-^L+\tilde{\gamma}_-^U}{2}=\frac{\sum_{\tilde{\gamma}\in\tilde{h}}\left(\tilde{\gamma}_-^L+\tilde{\gamma}_-^U\right)}{2l_{\tilde{h}}}\le\frac{\sum_{\tilde{\gamma}\in\tilde{h}}\left(\tilde{\gamma}^L+\tilde{\gamma}^U\right)}{2l_{\tilde{h}}}\le\frac{\sum_{\tilde{\gamma}\in\tilde{h}}\left(\tilde{\gamma}_+^L+\tilde{\gamma}_+^U\right)}{2l_{\tilde{h}}}=\frac{\tilde{\gamma}_+^L+\tilde{\gamma}_+^U}{2}$$

It follows that

$$s\left(\tilde{h}^-\right)\le s\left(\tilde{h}\right)\le s\left(\tilde{h}^+\right)$$

which implies $\tilde{h}^-\le \text{A-IVHFOWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)\le\tilde{h}^+$. This completes the proof. □

Theorem 3.4 implies that the aggregated value by using the A-IVHFOWA operator ranges between the biggest IVHFE and the smallest IVHFE.

Theorem 3.5. Let $\tilde{h}_i$ ($i=1,2,\cdots,n$) be a collection of IVHFEs and let $w=\left(w_1,w_2,\cdots,w_n\right)^T$ be the aggregation-associated vector satisfying $w_i\in[0,1]$ and $\sum_{i=1}^n w_i=1$, if $\tilde{h}$ is an IVHFN, then

$$\text{A-IVHFOWA}\left(\tilde{h}_1\oplus\tilde{h},\tilde{h}_2\oplus\tilde{h},\cdots,\tilde{h}_n\oplus\tilde{h}\right)=\text{A-IVHFOWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)\oplus\tilde{h} \tag{11}$$

Proof. For any $i=1,2,\cdots,n$, it follows from Definition 2.10 that

$$\tilde{h}_i\oplus\tilde{h}=\left\{\left[f^{-1}\left(f\left(\tilde{\gamma}_i^L\right)+f\left(\tilde{\gamma}^L\right)\right),f^{-1}\left(f\left(\tilde{\gamma}_i^U\right)+f\left(\tilde{\gamma}^U\right)\right)\right]\middle|\tilde{\gamma}_i\in\tilde{h}_i,\tilde{\gamma}\in\tilde{h}\right\}$$

Based on Theorem 3.1, we have

$$\begin{aligned}&\text{A-IVHFOWA}\left(\tilde{h}_1\oplus\tilde{h},\tilde{h}_2\oplus\tilde{h},\cdots,\tilde{h}_n\oplus\tilde{h}\right)\\&=\left\{\left[\begin{array}{l}f^{-1}\left(\sum_{i=1}^n w_i f\left(f^{-1}\left(f\left(\tilde{\gamma}_{\sigma(i)}^L\right)+f\left(\tilde{\gamma}^L\right)\right)\right)\right),\\ f^{-1}\left(\sum_{i=1}^n w_i f\left(f^{-1}\left(f\left(\tilde{\gamma}_{\sigma(i)}^U\right)+f\left(\tilde{\gamma}^U\right)\right)\right)\right)\end{array}\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\tilde{\gamma}_{\sigma(2)}\in\tilde{h}_{\sigma(2)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)},\tilde{\gamma}\in\tilde{h}\right\}\\&=\left\{\left[f^{-1}\left(\sum_{i=1}^n w_i f\left(\tilde{\gamma}_{\sigma(i)}^L\right)+f\left(\tilde{\gamma}^L\right)\right),f^{-1}\left(\sum_{i=1}^n w_i f\left(\tilde{\gamma}_{\sigma(i)}^U\right)+f\left(\tilde{\gamma}^U\right)\right)\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\tilde{\gamma}_{\sigma(2)}\in\tilde{h}_{\sigma(2)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)},\tilde{\gamma}\in\tilde{h}\right\}\end{aligned}$$

where $\tilde{h}_{\sigma(i)}$ is the *i*th largest of $\tilde{h}_i$ ($i=1,2,\cdots,n$), on the other hand, according to Definition 2.10, we can obtain

$$\begin{aligned}
&\text{A-IVHFOWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)\oplus\tilde{h}\\
&=\left\{\left[f^{-1}\left(\sum_{i=1}^n w_i f\left(\tilde{\gamma}_{\sigma(i)}^L\right)\right),f^{-1}\left(\sum_{i=1}^n w_i f\left(\tilde{\gamma}_{\sigma(i)}^U\right)\right)\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\tilde{\gamma}_{\sigma(2)}\in\tilde{h}_{\sigma(2)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\}\oplus\left\{\left[\tilde{\gamma}^L,\tilde{\gamma}^U\right]\middle|\tilde{\gamma}\in\tilde{h}\right\}\\
&=\left\{\begin{bmatrix}f^{-1}\left(f\left(f^{-1}\left(\sum_{i=1}^n w_i f\left(\tilde{\gamma}_{\sigma(i)}^L\right)\right)\right)+f\left(\tilde{\gamma}^L\right)\right),\\ f^{-1}\left(f\left(f^{-1}\left(\sum_{i=1}^n w_i f\left(\tilde{\gamma}_{\sigma(i)}^U\right)\right)\right)+f\left(\tilde{\gamma}^U\right)\right)\end{bmatrix}\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\tilde{\gamma}_{\sigma(2)}\in\tilde{h}_{\sigma(2)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)},\tilde{\gamma}\in\tilde{h}\right\}\\
&=\left\{\left[f^{-1}\left(\sum_{i=1}^n w_i f\left(\tilde{\gamma}_{\sigma(i)}^L\right)+f\left(\tilde{\gamma}^L\right)\right),f^{-1}\left(\sum_{i=1}^n w_i f\left(\tilde{\gamma}_{\sigma(i)}^U\right)+f\left(\tilde{\gamma}^U\right)\right)\right]\middle|\tilde{\gamma}_1\in\tilde{h}_1,\tilde{\gamma}_2\in\tilde{h}_2,\cdots,\tilde{\gamma}_n\in\tilde{h}_n,\tilde{\gamma}\in\tilde{h}\right\}
\end{aligned}$$

which indicates that Eq. (11) holds. This completes the proof. □

Theorem 3.6. Let $\tilde{h}_i$ ($i=1,2,\cdots,n$) be a collection of IVHFEs and let $w=\left(w_1,w_2,\cdots,w_n\right)^T$ be the aggregation-associated vector satisfying $w_i\in[0,1]$ and $\sum_{i=1}^n w_i=1$, and $r>0$. Then, we have

$$\text{A-IVHFOWA}\left(r\tilde{h}_1,r\tilde{h}_2,\cdots,r\tilde{h}_n\right)=r\text{A-IVHFOWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)\tag{12}$$

Proof. For any $i=1,2,\cdots,n$, it follows from Definition 2.10 that

$$r\tilde{h}_i=\left\{\left[f^{-1}\left(rf\left(\tilde{\gamma}_i^L\right)\right),f^{-1}\left(rf\left(\tilde{\gamma}_i^U\right)\right)\right]\middle|\tilde{\gamma}_i\in\tilde{h}_i\right\}$$

Based on Theorem 3.1 and Definition 2.10, we have

$$\begin{aligned}
&\text{A-IVHFOWA}\left(r\tilde{h}_1,r\tilde{h}_2,\cdots,r\tilde{h}_n\right)\\
&=\left\{\left[f^{-1}\left(\sum_{i=1}^n w_i f\left(f^{-1}\left(rf\left(\tilde{\gamma}_{\sigma(i)}^L\right)\right)\right)\right),f^{-1}\left(\sum_{i=1}^n w_i f\left(f^{-1}\left(rf\left(\tilde{\gamma}_{\sigma(i)}^U\right)\right)\right)\right)\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\tilde{\gamma}_{\sigma(2)}\in\tilde{h}_{\sigma(2)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\}\\
&=\left\{\left[f^{-1}\left(\sum_{i=1}^n w_i rf\left(\tilde{\gamma}_{\sigma(i)}^L\right)\right),f^{-1}\left(\sum_{i=1}^n w_i rf\left(\tilde{\gamma}_{\sigma(i)}^U\right)\right)\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\tilde{\gamma}_{\sigma(2)}\in\tilde{h}_{\sigma(2)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\}\\
&=\left\{\left[f^{-1}\left(r\sum_{i=1}^n w_i f\left(\tilde{\gamma}_{\sigma(i)}^L\right)\right),f^{-1}\left(r\sum_{i=1}^n w_i f\left(\tilde{\gamma}_{\sigma(i)}^U\right)\right)\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\tilde{\gamma}_{\sigma(2)}\in\tilde{h}_{\sigma(2)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\}
\end{aligned}$$

and

$$\begin{aligned}
&r\text{A-IVHFOWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)\\
&=r\left\{\left[f^{-1}\left(\sum_{i=1}^n w_i f\left(\tilde{\gamma}_{\sigma(i)}^L\right)\right),f^{-1}\left(\sum_{i=1}^n w_i f\left(\tilde{\gamma}_{\sigma(i)}^U\right)\right)\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\tilde{\gamma}_{\sigma(2)}\in\tilde{h}_{\sigma(2)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\}\\
&=\left\{\left[f^{-1}\left(rf\left(f^{-1}\left(\sum_{i=1}^n w_i f\left(\tilde{\gamma}_{\sigma(i)}^L\right)\right)\right)\right),f^{-1}\left(rf\left(f^{-1}\left(\sum_{i=1}^n w_i f\left(\tilde{\gamma}_{\sigma(i)}^U\right)\right)\right)\right)\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\tilde{\gamma}_{\sigma(2)}\in\tilde{h}_{\sigma(2)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\}\\
&=\left\{\left[f^{-1}\left(r\sum_{i=1}^n w_i f\left(\tilde{\gamma}_{\sigma(i)}^L\right)\right),f^{-1}\left(r\sum_{i=1}^n w_i f\left(\tilde{\gamma}_{\sigma(i)}^U\right)\right)\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\tilde{\gamma}_{\sigma(2)}\in\tilde{h}_{\sigma(2)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\}
\end{aligned}$$

where $\tilde{h}_{\sigma(i)}$ is the ith largest of $\tilde{h}_i$ ($i=1,2,\cdots,n$), which indicates that Eq. (12) holds. This completes the proof of Theorem 3.6. □

According to Theorems 3.5 and 3.6, we can easily obtain the following Theorem 3.7.

Theorem 3.7. Let $\tilde{h}_i$ ($i=1,2,\cdots,n$) be a collection of IVHFEs and let $w=\left(w_1,w_2,\cdots,w_n\right)^T$ be the aggregation-associated vector satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, if $r>0$, $\tilde{h}$ is an IVHFE, then

$$\text{A-IVHFOWA}\left(r\tilde{h}_1 \oplus \tilde{h}, r\tilde{h}_2 \oplus \tilde{h}, \cdots, r\tilde{h}_n \oplus \tilde{h}\right) = r\text{A-IVHFOWA}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right) \oplus \tilde{h} \quad (13)$$

In the following, we investigate some specific cases of the A-IVHFOWA operator under the assumption that the additive generator g is assigned different forms.

Case 1. If $g(t) = -\log(t)$, then the A-IVHFOWA operator is reduced to the IVHFOWA operator defined by Chen et al. [20]:

$$\text{IVHFOWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right) = \left\{\left[1-\prod_{i=1}^{n}\left(1-\tilde{\gamma}_{\sigma(i)}^{L}\right)^{w_i}, 1-\prod_{i=1}^{n}\left(1-\tilde{\gamma}_{\sigma(i)}^{U}\right)^{w_i}\right] \middle| \tilde{\gamma}_{\sigma(1)} \in \tilde{h}_{\sigma(1)},\cdots,\tilde{\gamma}_{\sigma(n)} \in \tilde{h}_{\sigma(n)}\right\} \quad (14)$$

Proof. If $g(t) = -\log(t)$, then $f(t) = g(1-t) = -\log(1-t)$ and $f^{-1}(t) = 1-e^{-t}$. Thus,

$$\begin{aligned}
&\text{IVHFOWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right) \\
&= \left\{\left[f^{-1}\left(\sum_{i=1}^{n} w_i f\left(\tilde{\gamma}_{\sigma(i)}^{L}\right)\right), f^{-1}\left(\sum_{i=1}^{n} w_i f\left(\tilde{\gamma}_{\sigma(i)}^{U}\right)\right)\right] \middle| \tilde{\gamma}_{\sigma(1)} \in \tilde{h}_{\sigma(1)}, \tilde{\gamma}_{\sigma(2)} \in \tilde{h}_{\sigma(2)},\cdots,\tilde{\gamma}_{\sigma(n)} \in \tilde{h}_{\sigma(n)}\right\} \\
&= \left\{\left[1-e^{\sum_{i=1}^{n} w_i \log\left(1-\tilde{\gamma}_{\sigma(i)}^{L}\right)}, 1-e^{\sum_{i=1}^{n} w_i \log\left(1-\tilde{\gamma}_{\sigma(i)}^{U}\right)}\right] \middle| \tilde{\gamma}_{\sigma(1)} \in \tilde{h}_{\sigma(1)}, \tilde{\gamma}_{\sigma(2)} \in \tilde{h}_{\sigma(2)},\cdots,\tilde{\gamma}_{\sigma(n)} \in \tilde{h}_{\sigma(n)}\right\} \\
&= \left\{\left[1-e^{\log\left(\prod_{i=1}^{n}\left(1-\tilde{\gamma}_{\sigma(i)}^{L}\right)^{w_i}\right)}, 1-e^{\log\left(\prod_{i=1}^{n}\left(1-\tilde{\gamma}_{\sigma(i)}^{U}\right)^{w_i}\right)}\right] \middle| \tilde{\gamma}_{\sigma(1)} \in \tilde{h}_{\sigma(1)}, \tilde{\gamma}_{\sigma(2)} \in \tilde{h}_{\sigma(2)},\cdots,\tilde{\gamma}_{\sigma(n)} \in \tilde{h}_{\sigma(n)}\right\} \\
&= \left\{\left[1-\prod_{i=1}^{n}\left(1-\tilde{\gamma}_{\sigma(i)}^{L}\right)^{w_i}, 1-\prod_{i=1}^{n}\left(1-\tilde{\gamma}_{\sigma(i)}^{U}\right)^{w_i}\right] \middle| \tilde{\gamma}_{\sigma(1)} \in \tilde{h}_{\sigma(1)}, \tilde{\gamma}_{\sigma(2)} \in \tilde{h}_{\sigma(2)},\cdots,\tilde{\gamma}_{\sigma(n)} \in \tilde{h}_{\sigma(n)}\right\}
\end{aligned}$$

Furthermore, if $\tilde{\gamma}_i^L = \tilde{\gamma}_i^U$ for any $\tilde{\gamma}_i \in \tilde{h}_i$ ($i=1,2,\cdots,n$), i.e., $\tilde{h}_i$ reduces to the HFEs $h_i = \bigcup_{\gamma_i \in h_i}\{\gamma_i\}$ ($i=1,2,\cdots,n$), then the Eq. (14) reduces to the HFOWA operator proposed by Xia and Xu [11]:

$$\text{HFOWA}\left(h_1,h_2,\cdots,h_n\right) = \bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)},\gamma_{\sigma(2)} \in h_{\sigma(2)},\cdots,\gamma_{\sigma(n)} \in h_{\sigma(n)}} \left\{1-\prod_{i=1}^{n}\left(1-\gamma_{\sigma(i)}\right)^{w_i}\right\} \quad (15)$$

Case 2. If $g(t)=\log\left(\frac{2-t}{t}\right)$, then the A-IVHFOWA operator reduces to the IVHFEOWA operator developed by Wei and Zhao [22]:

$$\text{IVHFEOWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=$$

$$\left\{\left[\frac{\prod_{i=1}^{n}\left(1+\tilde{\gamma}_{\sigma(i)}^{L}\right)^{w_i}-\prod_{i=1}^{n}\left(1-\tilde{\gamma}_{\sigma(i)}^{L}\right)^{w_i}}{\prod_{i=1}^{n}\left(1+\tilde{\gamma}_{\sigma(i)}^{L}\right)^{w_i}+\prod_{i=1}^{n}\left(1-\tilde{\gamma}_{\sigma(i)}^{L}\right)^{w_i}},\frac{\prod_{i=1}^{n}\left(1+\tilde{\gamma}_{\sigma(i)}^{U}\right)^{w_i}-\prod_{i=1}^{n}\left(1-\tilde{\gamma}_{\sigma(i)}^{U}\right)^{w_i}}{\prod_{i=1}^{n}\left(1+\tilde{\gamma}_{\sigma(i)}^{U}\right)^{w_i}+\prod_{i=1}^{n}\left(1-\tilde{\gamma}_{\sigma(i)}^{U}\right)^{w_i}}\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\} \quad (16)$$

Proof. If $g(t)=\log\left(\frac{2-t}{t}\right)$, then $f(t)=g(1-t)=\log\left(\frac{1+t}{1-t}\right)$ and $f^{-1}(t)=\frac{e^t-1}{e^t+1}$. Thus,

$$\text{IVHFEOWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)$$

$$=\left\{\left[f^{-1}\left(\sum_{i=1}^{n}w_i f\left(\tilde{\gamma}_{\sigma(i)}^{L}\right)\right),f^{-1}\left(\sum_{i=1}^{n}w_i f\left(\tilde{\gamma}_{\sigma(i)}^{U}\right)\right)\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\}$$

$$=\left\{\left[\frac{e^{\sum_{i=1}^{n}w_i\log\left(\frac{1+\tilde{\gamma}_{\sigma(i)}^{L}}{1-\tilde{\gamma}_{\sigma(i)}^{L}}\right)}-1}{e^{\sum_{i=1}^{n}w_i\log\left(\frac{1+\tilde{\gamma}_{\sigma(i)}^{L}}{1-\tilde{\gamma}_{\sigma(i)}^{L}}\right)}+1},\frac{e^{\sum_{i=1}^{n}w_i\log\left(\frac{1+\tilde{\gamma}_{\sigma(i)}^{U}}{1-\tilde{\gamma}_{\sigma(i)}^{U}}\right)}-1}{e^{\sum_{i=1}^{n}w_i\log\left(\frac{1+\tilde{\gamma}_{\sigma(i)}^{U}}{1-\tilde{\gamma}_{\sigma(i)}^{U}}\right)}+1}\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\}$$

$$=\left\{\left[\frac{e^{\log\left(\prod_{i=1}^{n}\left(\frac{1+\tilde{\gamma}_{\sigma(i)}^{L}}{1-\tilde{\gamma}_{\sigma(i)}^{L}}\right)^{w_i}\right)}-1}{e^{\log\left(\prod_{i=1}^{n}\left(\frac{1+\tilde{\gamma}_{\sigma(i)}^{L}}{1-\tilde{\gamma}_{\sigma(i)}^{L}}\right)^{w_i}\right)}+1},\frac{e^{\log\left(\prod_{i=1}^{n}\left(\frac{1+\tilde{\gamma}_{\sigma(i)}^{U}}{1-\tilde{\gamma}_{\sigma(i)}^{U}}\right)^{w_i}\right)}-1}{e^{\log\left(\prod_{i=1}^{n}\left(\frac{1+\tilde{\gamma}_{\sigma(i)}^{U}}{1-\tilde{\gamma}_{\sigma(i)}^{U}}\right)^{w_i}\right)}+1}\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\}$$

$$=\left\{\left[\frac{\prod_{i=1}^{n}\left(\frac{1+\tilde{\gamma}_{\sigma(i)}^{L}}{1-\tilde{\gamma}_{\sigma(i)}^{L}}\right)^{w_i}-1}{\prod_{i=1}^{n}\left(\frac{1+\tilde{\gamma}_{\sigma(i)}^{L}}{1-\tilde{\gamma}_{\sigma(i)}^{L}}\right)^{w_i}+1},\frac{\prod_{i=1}^{n}\left(\frac{1+\tilde{\gamma}_{\sigma(i)}^{U}}{1-\tilde{\gamma}_{\sigma(i)}^{U}}\right)^{w_i}-1}{\prod_{i=1}^{n}\left(\frac{1+\tilde{\gamma}_{\sigma(i)}^{U}}{1-\tilde{\gamma}_{\sigma(i)}^{U}}\right)^{w_i}+1}\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\}$$

$$=\left\{\left[\frac{\prod_{i=1}^{n}\left(1+\tilde{\gamma}_{\sigma(i)}^{L}\right)^{w_i}-\prod_{i=1}^{n}\left(1-\tilde{\gamma}_{\sigma(i)}^{L}\right)^{w_i}}{\prod_{i=1}^{n}\left(1+\tilde{\gamma}_{\sigma(i)}^{L}\right)^{w_i}+\prod_{i=1}^{n}\left(1-\tilde{\gamma}_{\sigma(i)}^{L}\right)^{w_i}},\frac{\prod_{i=1}^{n}\left(1+\tilde{\gamma}_{\sigma(i)}^{U}\right)^{w_i}-\prod_{i=1}^{n}\left(1-\tilde{\gamma}_{\sigma(i)}^{U}\right)^{w_i}}{\prod_{i=1}^{n}\left(1+\tilde{\gamma}_{\sigma(i)}^{U}\right)^{w_i}+\prod_{i=1}^{n}\left(1-\tilde{\gamma}_{\sigma(i)}^{U}\right)^{w_i}}\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\}$$

Furthermore, if $\tilde{\gamma}_i^L=\tilde{\gamma}_i^U$ for any $\tilde{\gamma}_i\in\tilde{h}_i$ ($i=1,2,\cdots,n$), i.e., $\tilde{h}_i$ reduces to the HFEs $h_i=\bigcup_{\gamma_i\in h_i}\{\gamma_i\}$ ($i=1,2,\cdots,n$), then the Eq. (16) is transformed into the hesitant fuzzy Einstein ordered weighted average (HFEOWA) operator proposed by Yu [15]:

$$\text{HFEOWA}\left(h_1,h_2,\cdots,h_n\right)=\bigcup_{\gamma_{\sigma(1)}\in h_{\sigma(1)},\gamma_{\sigma(2)}\in h_{\sigma(2)},\cdots,\gamma_{\sigma(n)}\in h_{\sigma(n)}}\left\{\frac{\prod_{i=1}^{n}\left(1+\gamma_{\sigma(i)}\right)^{w_i}-\prod_{i=1}^{n}\left(1-\gamma_{\sigma(i)}\right)^{w_i}}{\prod_{i=1}^{n}\left(1+\gamma_{\sigma(i)}\right)^{w_i}+\prod_{i=1}^{n}\left(1-\gamma_{\sigma(i)}\right)^{w_i}}\right\} \quad (17)$$

Case 3. If $g(t)=\log\left(\frac{\theta+(1-\theta)t}{t}\right)$, $\theta>0$, then the A-IVHFOWA operator is reduced to the IVHFHOWA operator:

$$\text{IVHFHOWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\left\{\left[\frac{\prod_{i=1}^{n}\left(1+(\theta-1)\tilde{\gamma}^L_{\sigma(i)}\right)^{w_i}-\prod_{i=1}^{n}\left(1-\tilde{\gamma}^L_{\sigma(i)}\right)^{w_i}}{\prod_{i=1}^{n}\left(1+(\theta-1)\tilde{\gamma}^L_{\sigma(i)}\right)^{w_i}+(\theta-1)\prod_{i=1}^{n}\left(1-\tilde{\gamma}^L_{\sigma(i)}\right)^{w_i}},\frac{\prod_{i=1}^{n}\left(1+(\theta-1)\tilde{\gamma}^U_{\sigma(i)}\right)^{w_i}-\prod_{i=1}^{n}\left(1-\tilde{\gamma}^U_{\sigma(i)}\right)^{w_i}}{\prod_{i=1}^{n}\left(1+(\theta-1)\tilde{\gamma}^U_{\sigma(i)}\right)^{w_i}+(\theta-1)\prod_{i=1}^{n}\left(1-\tilde{\gamma}^U_{\sigma(i)}\right)^{w_i}}\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\} \quad (18)$$

Proof. If $g(t)=\log\left(\frac{\theta+(1-\theta)t}{t}\right)$, then $f(t)=g(1-t)=\log\left(\frac{1+(\theta-1)t}{1-t}\right)$ and $f^{-1}(t)=\frac{1-e^t}{1-\theta-e^t}$. Thus,

$$\begin{aligned}
&\text{IVHFHOWA}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)\\
&=\left\{\left[f^{-1}\left(\sum_{i=1}^{n}w_i f\left(\tilde{\gamma}^L_{\sigma(i)}\right)\right),f^{-1}\left(\sum_{i=1}^{n}w_i f\left(\tilde{\gamma}^U_{\sigma(i)}\right)\right)\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\}\\
&=\left\{\left[\frac{1-e^{\log\left(\prod_{i=1}^{n}\left(\frac{1+(\theta-1)\tilde{\gamma}^L_{\sigma(i)}}{1-\tilde{\gamma}^L_{\sigma(i)}}\right)^{w_i}\right)}}{1-\theta-e^{\log\left(\prod_{i=1}^{n}\left(\frac{1+(\theta-1)\tilde{\gamma}^L_{\sigma(i)}}{1-\tilde{\gamma}^L_{\sigma(i)}}\right)^{w_i}\right)}},\frac{1-e^{\log\left(\prod_{i=1}^{n}\left(\frac{1+(\theta-1)\tilde{\gamma}^U_{\sigma(i)}}{1-\tilde{\gamma}^U_{\sigma(i)}}\right)^{w_i}\right)}}{1-\theta-e^{\log\left(\prod_{i=1}^{n}\left(\frac{1+(\theta-1)\tilde{\gamma}^U_{\sigma(i)}}{1-\tilde{\gamma}^U_{\sigma(i)}}\right)^{w_i}\right)}}\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\}\\
&=\left\{\left[\frac{\prod_{i=1}^{n}\left(1+(\theta-1)\tilde{\gamma}^L_{\sigma(i)}\right)^{w_i}-\prod_{i=1}^{n}\left(1-\tilde{\gamma}^L_{\sigma(i)}\right)^{w_i}}{\prod_{i=1}^{n}\left(1+(\theta-1)\tilde{\gamma}^L_{\sigma(i)}\right)^{w_i}+(\theta-1)\prod_{i=1}^{n}\left(1-\tilde{\gamma}^L_{\sigma(i)}\right)^{w_i}},\frac{\prod_{i=1}^{n}\left(1+(\theta-1)\tilde{\gamma}^U_{\sigma(i)}\right)^{w_i}-\prod_{i=1}^{n}\left(1-\tilde{\gamma}^U_{\sigma(i)}\right)^{w_i}}{\prod_{i=1}^{n}\left(1+(\theta-1)\tilde{\gamma}^U_{\sigma(i)}\right)^{w_i}+(\theta-1)\prod_{i=1}^{n}\left(1-\tilde{\gamma}^U_{\sigma(i)}\right)^{w_i}}\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\}
\end{aligned}$$

In particular, when $\theta=1$, the IVHFHOWA operator is degraded to the IVHFOWA operator (Eq. (14)); when $\theta=2$, the IVHFHOWA operator is degraded to the IVHFEOWA operator (Eq. (16)).

Furthermore, if $\tilde{\gamma}_i^L = \tilde{\gamma}_i^U$ for any $\tilde{\gamma}_i \in \tilde{h}_i$ ($i = 1, 2, \cdots, n$), i.e., $\tilde{h}_i$ reduces to the HFEs $h_i = \bigcup_{\gamma_i \in h_i} \{\gamma_i\}$ ($i = 1, 2, \cdots, n$), then the Eq. (18) is transformed into the hesitant fuzzy Hammer ordered weighted average (HFHOWA) operator given by Zhou et al. [16]:

$$\mathrm{HFHOWA}(h_1, h_2, \cdots, h_n) = \bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}} \left\{ \frac{\prod_{i=1}^{n} \left(1 + (\theta - 1)\gamma_{\sigma(i)}\right)^{w_i} - \prod_{i=1}^{n} \left(1 - \gamma_{\sigma(i)}\right)^{w_i}}{\prod_{i=1}^{n} \left(1 + (\theta - 1)\gamma_{\sigma(i)}\right)^{w_i} + (\theta - 1)\prod_{i=1}^{n} \left(1 - \gamma_{\sigma(i)}\right)^{w_i}} \right\} \quad (19)$$

In particular, when $\theta = 1$, the HFHOWA operator is degraded to the HFOWA operator (Eq. (15)); when $\theta = 2$, then the HFHOWA operator is degraded to the HFEOWA operator (Eq. (17)).

Case 4. If $g(t) = \log\left(\frac{\theta - 1}{\theta^t - 1}\right)$, $\theta > 1$, then the A-IVHFOWA operator is reduced to the interval-valued hesitant fuzzy Frank ordered weighted averaging (IVHFFOWA) operator:

$$\mathrm{IVHFFOWA}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right)$$
$$= \left\{ \left[1 - \log_\theta \left(1 + \prod_{i=1}^{n} \left(\theta^{1 - \tilde{\gamma}_{\sigma(i)}^L} - 1 \right)^{w_i} \right), 1 - \log_\theta \left(1 + \prod_{i=1}^{n} \left(\theta^{1 - \tilde{\gamma}_{\sigma(i)}^U} - 1 \right)^{w_i} \right) \right] \middle| \tilde{\gamma}_{\sigma(1)} \in \tilde{h}_{\sigma(1)}, \cdots, \tilde{\gamma}_{\sigma(n)} \in \tilde{h}_{\sigma(n)} \right\} \quad (20)$$

Proof. If $g(t) = \log\left(\frac{\theta - 1}{\theta^t - 1}\right)$, then $f(t) = g(1 - t) = \log\left(\frac{\theta - 1}{\theta^{1-t} - 1}\right)$ and $f^{-1}(t) = 1 - \log_\theta\left(\frac{\theta - 1 + e^t}{e^t}\right)$. Thus,

$$\mathrm{IVHFFOWA}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right)$$
$$= \left\{ \left[f^{-1}\left(\sum_{i=1}^{n} w_i f\left(\tilde{\gamma}_{\sigma(i)}^L\right) \right), f^{-1}\left(\sum_{i=1}^{n} w_i f\left(\tilde{\gamma}_{\sigma(i)}^U\right) \right) \right] \middle| \tilde{\gamma}_{\sigma(1)} \in \tilde{h}_{\sigma(1)}, \cdots, \tilde{\gamma}_{\sigma(n)} \in \tilde{h}_{\sigma(n)} \right\}$$
$$= \left\{ \left[1 - \log_\theta \left(\frac{\theta - 1 + e^{\log\left(\prod_{i=1}^{n}\left(\frac{\theta - 1}{\theta^{1-\tilde{\gamma}_{\sigma(i)}^L} - 1}\right)^{w_i}\right)}}{e^{\log\left(\prod_{i=1}^{n}\left(\frac{\theta - 1}{\theta^{1-\tilde{\gamma}_{\sigma(i)}^L} - 1}\right)^{w_i}\right)}} \right), 1 - \log_\theta \left(\frac{\theta - 1 + e^{\log\left(\prod_{i=1}^{n}\left(\frac{\theta - 1}{\theta^{1-\tilde{\gamma}_{\sigma(i)}^U} - 1}\right)^{w_i}\right)}}{e^{\log\left(\prod_{i=1}^{n}\left(\frac{\theta - 1}{\theta^{1-\tilde{\gamma}_{\sigma(i)}^U} - 1}\right)^{w_i}\right)}} \right) \right] \middle| \tilde{\gamma}_{\sigma(1)} \in \tilde{h}_{\sigma(1)}, \cdots, \tilde{\gamma}_{\sigma(n)} \in \tilde{h}_{\sigma(n)} \right\}$$
$$= \left\{ \left[1 - \log_\theta \left(\frac{\theta - 1 + \prod_{i=1}^{n}\left(\frac{\theta - 1}{\theta^{1-\tilde{\gamma}_{\sigma(i)}^L} - 1}\right)^{w_i}}{\prod_{i=1}^{n}\left(\frac{\theta - 1}{\theta^{1-\tilde{\gamma}_{\sigma(i)}^L} - 1}\right)^{w_i}} \right), 1 - \log_\theta \left(\frac{\theta - 1 + \prod_{i=1}^{n}\left(\frac{\theta - 1}{\theta^{1-\tilde{\gamma}_{\sigma(i)}^U} - 1}\right)^{w_i}}{\prod_{i=1}^{n}\left(\frac{\theta - 1}{\theta^{1-\tilde{\gamma}_{\sigma(i)}^U} - 1}\right)^{w_i}} \right) \right] \middle| \tilde{\gamma}_{\sigma(1)} \in \tilde{h}_{\sigma(1)}, \cdots, \tilde{\gamma}_{\sigma(n)} \in \tilde{h}_{\sigma(n)} \right\}$$
$$= \left\{ \left[1 - \log_\theta \left(1 + \prod_{i=1}^{n} \left(\theta^{1 - \tilde{\gamma}_{\sigma(i)}^L} - 1 \right)^{w_i} \right), 1 - \log_\theta \left(1 + \prod_{i=1}^{n} \left(\theta^{1 - \tilde{\gamma}_{\sigma(i)}^U} - 1 \right)^{w_i} \right) \right] \middle| \tilde{\gamma}_{\sigma(1)} \in \tilde{h}_{\sigma(1)}, \cdots, \tilde{\gamma}_{\sigma(n)} \in \tilde{h}_{\sigma(n)} \right\}$$

In particular, if $\theta \to 1$, then the IVHFFOWA operator is reduced to the IVHFOWA operator (Eq. (14)).

Furthermore, if $\tilde{\gamma}_i^L = \tilde{\gamma}_i^U$ for any $\tilde{\gamma}_i \in \tilde{h}_i$ ($i = 1, 2, \cdots, n$), i.e., $\tilde{h}_i$ reduces to the HFEs $h_i = \bigcup_{\gamma_i \in h_i} \{\gamma_i\}$ ($i = 1, 2, \cdots, n$), then the Eq. (20) is transformed into the hesitant fuzzy Frank ordered weighted average (HFFOWA) operator:

$$\text{HFFOWA}(h_1, h_2, \cdots, h_n) = \bigcup_{\gamma_{\sigma(1)} \in h_{\sigma(1)}, \gamma_{\sigma(2)} \in h_{\sigma(2)}, \cdots, \gamma_{\sigma(n)} \in h_{\sigma(n)}} \left\{1 - \log_\theta \left(1 + \prod_{i=1}^{n} \left(\theta^{1-\gamma_{\sigma(i)}} - 1\right)^{w_i}\right)\right\} \quad (21)$$

In particular, if $\theta \to 1$, then the HFFOWA operator is reduced to the HFOWA operator (Eq. (15)).

By combing the A-IVHFOWA operator with the geometric mean, we next define an Archimedean t-conorm- and t-norm-based interval-valued hesitant fuzzy ordered weighted geometric (A-IVHFOWG) operator:

Definition 3.2. Let $\tilde{h}_i$ ($i = 1, 2, \cdots, n$) be a collection of IVHFEs, $\tilde{h}_{\sigma(i)}$ be the *i*th largest of them, $w = (w_1, w_2, \cdots, w_n)^T$ be the aggregation-associated vector such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then an Archimedean t-conorm- and t-norm-based interval-valued hesitant fuzzy ordered weighted geometric (A-IVHFOWG) operator is a mapping $\tilde{H}^n \to \tilde{H}$, such that

$$\text{A-IVHFOWG}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right) = \bigotimes_{i=1}^{n} \left(\tilde{h}_{\sigma(i)}^{w_i}\right) \quad (22)$$

Theorem 3.8. Let $\tilde{h}_i$ ($i = 1, 2, \cdots, n$) be a collection of IVHFEs, $\tilde{h}_{\sigma(i)}$ be the *i*th largest of them, and let $w = (w_1, w_2, \cdots, w_n)^T$ be the aggregation-associated vector satisfying $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$; then, the aggregated value by using the A-IVHFOWG operator is also an IVHFE, and

$$\text{A-IVHFOWG}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right) = \left\{\left[g^{-1}\left(\sum_{i=1}^{n} w_i g\left(\tilde{\gamma}_{\sigma(i)}^L\right)\right), g^{-1}\left(\sum_{i=1}^{n} w_i g\left(\tilde{\gamma}_{\sigma(i)}^U\right)\right)\right] \middle| \tilde{\gamma}_{\sigma(1)} \in \tilde{h}_{\sigma(1)}, \tilde{\gamma}_{\sigma(2)} \in \tilde{h}_{\sigma(2)}, \cdots, \tilde{\gamma}_{\sigma(n)} \in \tilde{h}_{\sigma(n)}\right\} \quad (23)$$

Example 3.2. Suppose that $\tilde{h}_1$, $\tilde{h}_2$, $\tilde{h}_3$, w, and $g(t)$ are shown as Example 3.1. Then, by Eq. (23), we can obtain

$$\text{A-IVHFOWG}\left(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3\right) = \left\{\begin{array}{l}[0.6640, 0.8172], [0.2591, 0.4209], [0.5521, 0.6854], [0.2569, 0.4071], [0.3572, 0.4885], [0.2449, 0.3683], \\ [0.4488, 0.6889], [0.2528, 0.4075], [0.4188, 0.6085], [0.2509, 0.3955], [0.3274, 0.4643], [0.2403, 0.3608]\end{array}\right\}$$

The A-IVHFOWG operator has some desirable characteristics similar to the A-IVHFOWA operator as follows. In should be noted that the proof of these characteristics are also similar to A-IVHFOWA. Therefore, we just list out these properties.

Theorem 3.9 (Commutativity). Let $\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n$ be a collection of IVHFEs. If $\tilde{h}_1', \tilde{h}_2', \cdots, \tilde{h}_n'$ is any permutation of $\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n$, then we have

$$\text{A-IVHFOWG}\left(\tilde{h}_1', \tilde{h}_2', \cdots, \tilde{h}_n'\right) = \text{A-IVHFOWG}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right) \quad (24)$$

Theorem 3.10 (Idempotency). Let $\tilde{h}_i$ ($i = 1, 2, \cdots, n$) be a collection of IVHFEs. If all $\tilde{h}_i$ ($i = 1, 2, \cdots, n$) are equal, i.e., $\tilde{h}_i = \tilde{h} = \left\{\tilde{\gamma} \middle| \tilde{\gamma} \in \tilde{h}\right\} = \left\{\left[\tilde{\gamma}^L, \tilde{\gamma}^U\right] \middle| \tilde{\gamma} \in \tilde{h}\right\}$, for all i, then

$$\text{A-IVHFOWG}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right) = \text{A-IVHFOWG}\left(\tilde{h}, \tilde{h}, \cdots, \tilde{h}\right) = \tilde{h} \quad (25)$$

Theorem 3.11 (Boundedness). For a collection of IVHFEs $\tilde{h}_i = \left\{\tilde{\gamma}_i \middle| \tilde{\gamma}_i \in \tilde{h}_i\right\} = \left\{\left[\tilde{\gamma}_i^L, \tilde{\gamma}_i^U\right] \middle| \tilde{\gamma}_i \in \tilde{h}_i\right\}$ ($i = 1, 2, \cdots, n$), let $\tilde{h}^- = \left\{\left[\min_{i=1}^{n} \min_{\tilde{\gamma}_i \in \tilde{h}_i} \tilde{\gamma}_i^L, \min_{i=1}^{n} \min_{\tilde{\gamma}_i \in \tilde{h}_i} \tilde{\gamma}_i^U\right]\right\}$, $\tilde{h}^+ = \left\{\left[\max_{i=1}^{n} \max_{\tilde{\gamma}_i \in \tilde{h}_i} \tilde{\gamma}_i^L, \max_{i=1}^{n} \max_{\tilde{\gamma}_i \in \tilde{h}_i} \tilde{\gamma}_i^U\right]\right\}$, then,

$$\tilde{h}^- \le \text{A-IVHFOWG}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right) \le \tilde{h}^+ \quad (26)$$

Theorem 3.12. Let $\tilde{h}_i$ ($i = 1, 2, \cdots, n$) be a collection of IVHFEs and let $w = \left(w_1, w_2, \cdots, w_n\right)^T$ be the aggregation-associated vector that satisfies $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. If $\tilde{h}$ is an IVHFN, then

$$\text{A-IVHFOWG}\left(\tilde{h}_1 \otimes \tilde{h}, \tilde{h}_2 \otimes \tilde{h}, \cdots, \tilde{h}_n \otimes \tilde{h}\right) = \text{A-IVHFOWG}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right) \otimes \tilde{h} \quad (27)$$

Theorem 3.13. Let $\tilde{h}_i$ ($i = 1, 2, \cdots, n$) be a collection of IVHFEs and let $w = \left(w_1, w_2, \cdots, w_n\right)^T$ be the aggregation-associated vector such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, and $r > 0$. Then, we have

$$\text{A-IVHFOWG}\left(\tilde{h}_1^r, \tilde{h}_2^r, \cdots, \tilde{h}_n^r\right) = \left(\text{A-IVHFOWG}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right)\right)^r \quad (28)$$

Theorem 3.14. Let $\tilde{h}_i$ ($i = 1, 2, \cdots, n$) be a collections of IVHFEs and let $w = \left(w_1, w_2, \cdots, w_n\right)^T$ be the aggregation-associated vector such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. If $r > 0$ and $\tilde{h}$ is an IVHFE, then

$$\text{A-IVHFOWG}\left(\tilde{h}_1^r \otimes \tilde{h}, \tilde{h}_2^r \otimes \tilde{h}, \cdots, \tilde{h}_n^r \otimes \tilde{h}\right) = \left(\text{A-IVHFOWG}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right)\right)^r \otimes \tilde{h} \quad (29)$$

In the following, we will investigate the relationship between the A-IVHFOWA and A-IVHFOWG operators.

Theorem 3.15. Let $\tilde{h}_i$ ($i = 1, 2, \cdots, n$) be a collections of IVHFEs and let $w = (w_1, w_2, \cdots, w_n)^T$ be the aggregation-associated vector such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. We then have the following:

(1) $\text{A-IVHFOWA}\left(\tilde{h}_1^c, \tilde{h}_2^c, \cdots, \tilde{h}_n^c\right) = \left(\text{A-IVHFOWG}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right)\right)^c$

(2) $\text{A-IVHFOWG}\left(\tilde{h}_1^c, \tilde{h}_2^c, \cdots, \tilde{h}_n^c\right) = \left(\text{A-IVHFOWA}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right)\right)^c$

Proof. (1) According to Eqs. (4) and (23), we can obtain

$$\begin{aligned}
&\text{A-IVHFOWA}\left(\tilde{h}_1^c, \tilde{h}_2^c, \cdots, \tilde{h}_n^c\right) \\
&= \left\{\left[f^{-1}\left(\sum_{i=1}^{n} w_i f\left(1-\tilde{\gamma}_{\sigma(i)}^U\right)\right), f^{-1}\left(\sum_{i=1}^{n} w_i f\left(1-\tilde{\gamma}_{\sigma(i)}^L\right)\right)\right] \middle| \tilde{\gamma}_{\sigma(1)} \in \tilde{h}_{\sigma(1)}, \tilde{\gamma}_{\sigma(2)} \in \tilde{h}_{\sigma(2)}, \cdots, \tilde{\gamma}_{\sigma(n)} \in \tilde{h}_{\sigma(n)}\right\} \\
&= \left\{\left[f^{-1}\left(\sum_{i=1}^{n} w_i g\left(\tilde{\gamma}_{\sigma(i)}^U\right)\right), f^{-1}\left(\sum_{i=1}^{n} w_i g\left(\tilde{\gamma}_{\sigma(i)}^L\right)\right)\right] \middle| \tilde{\gamma}_{\sigma(1)} \in \tilde{h}_{\sigma(1)}, \tilde{\gamma}_{\sigma(2)} \in \tilde{h}_{\sigma(2)}, \cdots, \tilde{\gamma}_{\sigma(n)} \in \tilde{h}_{\sigma(n)}\right\} \\
&= \left\{\left[1-g^{-1}\left(\sum_{i=1}^{n} w_i g\left(\tilde{\gamma}_{\sigma(i)}^U\right)\right), 1-g^{-1}\left(\sum_{i=1}^{n} w_i g\left(\tilde{\gamma}_{\sigma(i)}^L\right)\right)\right] \middle| \tilde{\gamma}_{\sigma(1)} \in \tilde{h}_{\sigma(1)}, \tilde{\gamma}_{\sigma(2)} \in \tilde{h}_{\sigma(2)}, \cdots, \tilde{\gamma}_{\sigma(n)} \in \tilde{h}_{\sigma(n)}\right\} \\
&= \left(\left\{\left[g^{-1}\left(\sum_{i=1}^{n} w_i g\left(\tilde{\gamma}_{\sigma(i)}^L\right)\right), g^{-1}\left(\sum_{i=1}^{n} w_i g\left(\tilde{\gamma}_{\sigma(i)}^U\right)\right)\right] \middle| \tilde{\gamma}_{\sigma(1)} \in \tilde{h}_{\sigma(1)}, \tilde{\gamma}_{\sigma(2)} \in \tilde{h}_{\sigma(2)}, \cdots, \tilde{\gamma}_{\sigma(n)} \in \tilde{h}_{\sigma(n)}\right\}\right)^c \\
&= \left(\text{A-IVHFOWG}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right)\right)^c
\end{aligned}$$

(2) According to Eqs. (4) and (23), we have

$$\begin{aligned}
&\text{A-IVHFOWG}\left(\tilde{h}_1^c, \tilde{h}_2^c, \cdots, \tilde{h}_n^c\right) \\
&= \left\{\left[g^{-1}\left(\sum_{i=1}^{n} w_i g\left(1-\tilde{\gamma}_{\sigma(i)}^U\right)\right), g^{-1}\left(\sum_{i=1}^{n} w_i g\left(1-\tilde{\gamma}_{\sigma(i)}^L\right)\right)\right] \middle| \tilde{\gamma}_{\sigma(1)} \in \tilde{h}_{\sigma(1)}, \tilde{\gamma}_{\sigma(2)} \in \tilde{h}_{\sigma(2)}, \cdots, \tilde{\gamma}_{\sigma(n)} \in \tilde{h}_{\sigma(n)}\right\} \\
&= \left\{\left[g^{-1}\left(\sum_{i=1}^{n} w_i f\left(\tilde{\gamma}_{\sigma(i)}^U\right)\right), g^{-1}\left(\sum_{i=1}^{n} w_i f\left(\tilde{\gamma}_{\sigma(i)}^L\right)\right)\right] \middle| \tilde{\gamma}_{\sigma(1)} \in \tilde{h}_{\sigma(1)}, \tilde{\gamma}_{\sigma(2)} \in \tilde{h}_{\sigma(2)}, \cdots, \tilde{\gamma}_{\sigma(n)} \in \tilde{h}_{\sigma(n)}\right\} \\
&= \left\{\left[1-f^{-1}\left(\sum_{i=1}^{n} w_i f\left(\tilde{\gamma}_{\sigma(i)}^U\right)\right), 1-f^{-1}\left(\sum_{i=1}^{n} w_i f\left(\tilde{\gamma}_{\sigma(i)}^L\right)\right)\right] \middle| \tilde{\gamma}_{\sigma(1)} \in \tilde{h}_{\sigma(1)}, \tilde{\gamma}_{\sigma(2)} \in \tilde{h}_{\sigma(2)}, \cdots, \tilde{\gamma}_{\sigma(n)} \in \tilde{h}_{\sigma(n)}\right\} \\
&= \left(\left\{\left[f^{-1}\left(\sum_{i=1}^{n} w_i f\left(\tilde{\gamma}_{\sigma(i)}^L\right)\right), f^{-1}\left(\sum_{i=1}^{n} w_i f\left(\tilde{\gamma}_{\sigma(i)}^U\right)\right)\right] \middle| \tilde{\gamma}_{\sigma(1)} \in \tilde{h}_{\sigma(1)}, \tilde{\gamma}_{\sigma(2)} \in \tilde{h}_{\sigma(2)}, \cdots, \tilde{\gamma}_{\sigma(n)} \in \tilde{h}_{\sigma(n)}\right\}\right)^c \\
&= \left(\text{A-IVHFOWA}\left(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n\right)\right)^c
\end{aligned}$$

The proof of Theorem 3.15 is completed. □

In what follows, we investigate some specific cases of the A-IVHFOWG operator under the assumption that the additive generator g is assigned different forms.

Case 1. If $g(t) = -\log(t)$, then the A-IVHFOWG operator is reduced to the IVHFOWG operator defined by Chen et al. [20]:

$$\text{IVHFOWG}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\left\{\left[\prod_{i=1}^{n}\left(\tilde{\gamma}_{\sigma(i)}^{L}\right)^{w_i},\prod_{i=1}^{n}\left(\tilde{\gamma}_{\sigma(i)}^{U}\right)^{w_i}\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\} \quad (30)$$

Furthermore, if $\tilde{\gamma}_i^L = \tilde{\gamma}_i^U$ for any $\tilde{\gamma}_i \in \tilde{h}_i$ ($i = 1, 2, \cdots, n$), i.e., $\tilde{h}_i$ reduces to the HFEs $h_i = \bigcup_{\gamma_i \in h_i}\{\gamma_i\}$ ($i = 1, 2, \cdots, n$), then the Eq. (30) reduces to the HFOWG operator proposed by Xia and Xu [11]:

$$\text{HFOWG}\left(h_1,h_2,\cdots,h_n\right)=\bigcup_{\gamma_{\sigma(1)}\in h_{\sigma(1)},\gamma_{\sigma(2)}\in h_{\sigma(2)},\cdots,\gamma_{\sigma(n)}\in h_{\sigma(n)}}\left\{\prod_{i=1}^{n}\gamma_{\sigma(i)}^{w_i}\right\} \quad (31)$$

Case 2. If $g(t) = \log\left(\frac{2-t}{t}\right)$, then the A-IVHFOWG operator is reduced to the IVHFEOWG operator proposed by Wei and Zhao [22]:

$$\begin{aligned}&\text{IVHFEOWG}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)\\&=\left\{\left[\frac{2\prod_{i=1}^{n}\left(\tilde{\gamma}_{\sigma(i)}^{L}\right)^{w_i}}{\prod_{i=1}^{n}\left(2-\tilde{\gamma}_{\sigma(i)}^{L}\right)^{w_i}+\prod_{i=1}^{n}\left(\tilde{\gamma}_{\sigma(i)}^{L}\right)^{w_i}},\frac{2\prod_{i=1}^{n}\left(\tilde{\gamma}_{\sigma(i)}^{U}\right)^{w_i}}{\prod_{i=1}^{n}\left(2-\tilde{\gamma}_{\sigma(i)}^{U}\right)^{w_i}+\prod_{i=1}^{n}\left(\tilde{\gamma}_{\sigma(i)}^{U}\right)^{w_i}}\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\}\end{aligned} \quad (32)$$

Furthermore, if $\tilde{\gamma}_i^L = \tilde{\gamma}_i^U$ for any $\tilde{\gamma}_i \in \tilde{h}_i$ ($i = 1, 2, \cdots, n$), i.e., $\tilde{h}_i$ reduces to the HFEs $h_i = \bigcup_{\gamma_i \in h_i}\{\gamma_i\}$ ($i = 1, 2, \cdots, n$), then the Eq. (32) is transformed into the HFEOWG operator proposed by Yu [15]:

$$\text{HFEOWG}\left(h_1,h_2,\cdots,h_n\right)=\bigcup_{\gamma_{\sigma(1)}\in h_{\sigma(1)},\gamma_{\sigma(2)}\in h_{\sigma(2)},\cdots,\gamma_{\sigma(n)}\in h_{\sigma(n)}}\left\{\frac{2\prod_{i=1}^{n}\gamma_{\sigma(i)}^{w_i}}{\prod_{i=1}^{n}\left(2-\gamma_{\sigma(i)}\right)^{w_i}+\prod_{i=1}^{n}\gamma_{\sigma(i)}^{w_i}}\right\} \quad (33)$$

Case 3. If $g(t) = \log\left(\frac{\theta+(1-\theta)t}{t}\right)$, $\theta > 0$, then the A-IVHFOWG operator is reduced to the interval-valued hesitant fuzzy Hammer ordered weighted geometric (IVHFHOWG) operator:

$$\mathrm{IVHFHOWG}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\left\{\left[\begin{array}{c}\dfrac{\theta\prod_{i=1}^{n}\left(\tilde{\gamma}_{\sigma(i)}^{L}\right)^{w_i}}{\prod_{i=1}^{n}\left(1+(\theta-1)\left(1-\tilde{\gamma}_{\sigma(i)}^{L}\right)\right)^{w_i}+(\theta-1)\prod_{i=1}^{n}\left(\tilde{\gamma}_{\sigma(i)}^{L}\right)^{w_i}},\\ \dfrac{\theta\prod_{i=1}^{n}\left(\tilde{\gamma}_{\sigma(i)}^{U}\right)^{w_i}}{\prod_{i=1}^{n}\left(1+(\theta-1)\left(1-\tilde{\gamma}_{\sigma(i)}^{U}\right)\right)^{w_i}+(\theta-1)\prod_{i=1}^{n}\left(\tilde{\gamma}_{\sigma(i)}^{U}\right)^{w_i}}\end{array}\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\} \quad (34)$$

In particular, if $\theta=1$, then the IVHFHOWG operator is degraded to the IVHFOWG operator (Eq. (30)); if $\theta=2$, then the IVHFHOWG operator is degraded to the IVHFEOWG operator (Eq. (32)).

Furthermore, if $\tilde{\gamma}_i^L=\tilde{\gamma}_i^U$ for any $\tilde{\gamma}_i\in\tilde{h}_i$ ($i=1,2,\cdots,n$), i.e., $\tilde{h}_i$ reduces to the HFEs $h_i=\bigcup_{\gamma_i\in h_i}\{\gamma_i\}$ ($i=1,2,\cdots,n$), then the Eq. (34) is transformed into the HFHOWG operator given by Zhou et al. [16]:

$$\mathrm{HFHOWG}\left(h_1,h_2,\cdots,h_n\right)=\bigcup_{\gamma_{\sigma(1)}\in h_{\sigma(1)},\gamma_{\sigma(2)}\in h_{\sigma(2)},\cdots,\gamma_{\sigma(n)}\in h_{\sigma(n)}}\left\{\frac{\theta\prod_{i=1}^{n}\gamma_{\sigma(i)}^{w_i}}{\prod_{i=1}^{n}\left(1+(\theta-1)\left(1-\gamma_{\sigma(i)}\right)\right)^{w_i}+(\theta-1)\prod_{i=1}^{n}\gamma_{\sigma(i)}^{w_i}}\right\} \quad (35)$$

In particular, when $\theta=1$, the HFHOWG operator is degraded to the HFOWG operator (Eq. (31)); when $\theta=2$, then the HFHOWG operator is degraded to the HFEOWG operator (Eq. (33)).

Case 4. If $g(t)=\log\left(\dfrac{\theta-1}{\theta^t-1}\right)$, $\theta>1$, then the A-IVHFOWG operator is reduced to the interval-valued hesitant fuzzy Frank ordered weighted geometric (IVHFFOWG) operator:

$$\begin{aligned}&\mathrm{IVHFFOWG}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)\\&=\left\{\left[\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{\tilde{\gamma}_{\sigma(i)}^{L}}-1\right)^{w_i}\right),\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{\tilde{\gamma}_{\sigma(i)}^{U}}-1\right)^{w_i}\right)\right]\middle|\tilde{\gamma}_{\sigma(1)}\in\tilde{h}_{\sigma(1)},\cdots,\tilde{\gamma}_{\sigma(n)}\in\tilde{h}_{\sigma(n)}\right\}\end{aligned} \quad (36)$$

In particular, if $\theta\to1$, then the IVHFFOWG operator is reduced to the IVHFOWG operator (Eq. (30)).

Furthermore, if $\tilde{\gamma}_i^L=\tilde{\gamma}_i^U$ for any $\tilde{\gamma}_i\in\tilde{h}_i$ ($i=1,2,\cdots,n$), i.e., $\tilde{h}_i$ reduces to the HFEs $h_i=\bigcup_{\gamma_i\in h_i}\{\gamma_i\}$ ($i=1,2,\cdots,n$), then the Eq. (36) is transformed into the hesitant fuzzy Frank ordered weighted geometric (HFFOWG) operator:

$$\mathrm{HFFOWG}\left(\tilde{h}_1,\tilde{h}_2,\cdots,\tilde{h}_n\right)=\bigcup_{\gamma_{\sigma(1)}\in h_{\sigma(1)},\gamma_{\sigma(2)}\in h_{\sigma(2)},\cdots,\gamma_{\sigma(n)}\in h_{\sigma(n)}}\left\{\log_\theta\left(1+\prod_{i=1}^{n}\left(\theta^{\gamma_{\sigma(i)}}-1\right)^{w_i}\right)\right\} \quad (37)$$

In particular, if $\theta\to1$, then the HFFOWG operator reduces to the HFOWG operator (Eq. (31)).

4 A Method for Multi-criteria Decision Making with Interval-valued Hesitant Fuzzy Information

In the following, we will use the proposed operators to propose a method for multi-criteria decision making (MCDM) within the context of IVHFSs. Let $Y=\{Y_1,Y_2,\cdots,Y_m\}$ be a set of m alternatives, and let $G=\{G_1,G_2,\ldots,G_n\}$ be a collection of n criteria whose weight vector is $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$, with $\omega_j\in[0,1]$, $j=1,2,\cdots,n$, and $\sum_{j=1}^{n}\omega_j=1$, where ω_j denotes the importance degree of the criterion G_j. The decision makers provide all the possible interval values for the alternative Y_i with respect to the criterion G_j, denoted by an IVHFE $\tilde{r}_{ij}=\{\tilde{\gamma}_{ij}\,|\,\tilde{\gamma}_{ij}\in\tilde{r}_{ij}\}=\{[\tilde{\gamma}_{ij}^L,\tilde{\gamma}_{ij}^U]\,\|\,\tilde{\gamma}_{ij}\in\tilde{r}_{ij}\}$. All $\tilde{r}_{ij}$ ($i=1,2,\cdots,m$; $j=1,2,\cdots,n$) constitute an interval-valued hesitant fuzzy decision matrix $\tilde{R}=(\tilde{r}_{ij})_{m\times n}$, which is shown in Table 1.

Table 1. Interval-valued hesitant fuzzy decision matrix $\tilde{R}$

	G_1	$\cdots$	G_j	$\cdots$	G_n
Y_1	$\tilde{r}_{11}$	$\cdots$	$\tilde{r}_{1j}$	$\cdots$	$\tilde{r}_{1n}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
Y_i	$\tilde{r}_{i1}$	$\cdots$	$\tilde{r}_{ij}$	$\cdots$	$\tilde{r}_{in}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
Y_m	$\tilde{r}_{m1}$	$\cdots$	$\tilde{r}_{mj}$	$\cdots$	$\tilde{r}_{mn}$

In general, there are benefit criteria (i.e., the bigger the criterion values, the better) and cost criteria (i.e., the smaller the criterion values, the better) in a MCDM problem. In such cases, we need transform the cost criteria into benefit criteria, i.e., use the method in [36] to transform the interval-valued hesitant fuzzy decision matrix $\tilde{R}=(\tilde{r}_{ij})_{m\times n}$ into a normalized interval-valued hesitant fuzzy decision matrix $\tilde{A}=(\tilde{a}_{ij})_{m\times n}$, where

$$\tilde{a}_{ij}=\begin{cases}\tilde{r}_{ij}, & \text{for benefit criterion } G_j\\ \tilde{r}_{ij}^c, & \text{for cost criterion } G_j\end{cases},\quad i=1,2,\cdots,m,\ j=1,2,\cdots,n \tag{38}$$

where $\tilde{r}_{ij}^c$ is the complement of $\tilde{r}_{ij}$ such that $\tilde{r}_{ij}^c=\{[1-\tilde{\gamma}_{ij}^U,1-\tilde{\gamma}_{ij}^L]\,\|\,\tilde{\gamma}_{ij}\in\tilde{r}_{ij}\}$.

Step 1. Transform the interval-valued hesitant fuzzy decision matrix $\tilde{R}=(\tilde{r}_{ij})_{m\times n}$ into the normalized interval-valued hesitant fuzzy decision matrix $\tilde{A}=(\tilde{a}_{ij})_{m\times n}$ on the basis of Eq. (38).

Step 2. Utilize the A-IVHFOWA operator

$$\tilde{a}_i = \text{A-IVHFOWA}\left(\tilde{a}_{i1}, \tilde{a}_{i2}, \cdots, \tilde{a}_{in}\right) = \bigoplus_{j=1}^{n}\left(w_j \tilde{a}_{i\sigma(j)}\right)$$
$$= \left\{ \left[f^{-1}\left(\sum_{j=1}^{n} w_j f\left(\tilde{\gamma}^{L}_{i\sigma(j)}\right)\right), f^{-1}\left(\sum_{j=1}^{n} w_j f\left(\tilde{\gamma}^{U}_{i\sigma(j)}\right)\right) \right] \middle| \tilde{\gamma}_{i\sigma(1)} \in \tilde{a}_{i\sigma(1)}, \tilde{\gamma}_{i\sigma(2)} \in \tilde{a}_{i\sigma(2)}, \cdots, \tilde{\gamma}_{i\sigma(n)} \in \tilde{a}_{i\sigma(n)} \right\}$$
$$i = 1, 2, \cdots, m \tag{39}$$

or the A-IVHFOWG operator

$$\tilde{a}_i = \text{A-IVHFOWG}\left(\tilde{a}_{i1}, \tilde{a}_{i2}, \cdots, \tilde{a}_{in}\right) = \bigoplus_{j=1}^{n}\left(w_j \tilde{a}_{i\sigma(j)}\right)$$
$$= \left\{ \left[g^{-1}\left(\sum_{j=1}^{n} w_j g\left(\tilde{\gamma}^{L}_{i\sigma(j)}\right)\right), g^{-1}\left(\sum_{j=1}^{n} w_j g\left(\tilde{\gamma}^{U}_{i\sigma(j)}\right)\right) \right] \middle| \tilde{\gamma}_{i\sigma(1)} \in \tilde{a}_{i\sigma(1)}, \tilde{\gamma}_{i\sigma(2)} \in \tilde{a}_{i\sigma(2)}, \cdots, \tilde{\gamma}_{i\sigma(n)} \in \tilde{a}_{i\sigma(n)} \right\}$$
$$i = 1, 2, \cdots, m \tag{40}$$

to fuse all of the performance values $\tilde{a}_{ij}$ ($j = 1, 2, \cdots, n$) in the ith line of $\tilde{A}$ and then derive the overall performance value $\tilde{a}_i$ ($i = 1, 2, \cdots, m$) of the alternative Y_i ($i = 1, 2, \cdots, m$), where $\left(\tilde{a}_{i\sigma(1)}, \tilde{a}_{i\sigma(2)}, \cdots, \tilde{a}_{i\sigma(n)}\right)$ is a permutation of $\left(\tilde{a}_{i1}, \tilde{a}_{i2}, \cdots, \tilde{a}_{in}\right)$, such that $\tilde{a}_{i\sigma(1)} \geq \tilde{a}_{i\sigma(2)} \geq \cdots \geq \tilde{a}_{i\sigma(n)}$, and $w = \left(w_1, w_2, \cdots, w_n\right)^T$ is the weight vector of the ordered positions of $\left(\tilde{a}_{i\sigma(1)}, \tilde{a}_{i\sigma(2)}, \cdots, \tilde{a}_{i\sigma(n)}\right)$, with $w_j \in [0,1]$, $j = 1, 2, \cdots, n$, and $\sum_{j=1}^{n} w_j = 1$.

Step 3. According to Definitions 2.3 and 2.4, calculate the score functions $s\left(\tilde{a}_i\right)$ and variance functions $v\left(\tilde{a}_i\right)$ of $\tilde{a}_i$ ($i = 1, 2, \cdots, m$), and then rank all of the alternatives Y_i ($i = 1, 2, \cdots, m$) in descending order as per Definition 2.5.

5 Illustrative Example

5.1 An illustrative example

In this subsection, a practical example (adapted from Herrera and Herrera-Viedma [37] is used to implement the developed method.

Example 5.1. An investment company wants to invest a sum of money in the best option. There is a panel with five possible alternatives to invest the money: (1) Y_1 is a car company; (2) Y_2 is a food company; (3) Y_3 is a computer company; (4) Y_4 is an arms company; (5) Y_5 is a TV company. The investment company must take a decision according to the following four attributes: (1) G_1 is the risk analysis; (2) G_2 is the growth analysis; (3) G_3 is the social-political impact analysis; (4) G_4 is the environmental impact analysis.

The weight vector of the four criteria G_j ($j=1,2,3,4$) is $\omega=(0.1,0.3,0.5,0.1)^T$. Assume that the performance of the alternatives Y_i ($i=1,2,3,4,5$) with respect to the criteria G_j ($j=1,2,3,4$) is denoted by an IVHFE $\tilde{r}_{ij}=\left\{\tilde{\gamma}_{ij}\middle|\tilde{\gamma}_{ij}\in\tilde{r}_{ij}\right\}=\left\{\left[\tilde{\gamma}_{ij}^L,\tilde{\gamma}_{ij}^U\right]\middle|\tilde{\gamma}_{ij}\in\tilde{r}_{ij}\right\}$, where $\tilde{\gamma}_{ij}=\left[\tilde{\gamma}_{ij}^L,\tilde{\gamma}_{ij}^U\right]$ indicates the possible interval values to which the alternative Y_i satisfies the criterion G_j . All of $\tilde{r}_{ij}$ ($i=1,2,3,4,5$; $j=1,2,3,4$) are contained in the interval-valued hesitant fuzzy decision matrix $\tilde{R}=\left(\tilde{r}_{ij}\right)_{5\times 4}$ (see Table 2).

Table 2. The interval-valued hesitant fuzzy decision matrix $\tilde{R}$

	G_1	G_2	G_3	G_4
Y_1	{[0.3, 0.5], [0.3, 0.4], [0.2, 0.3]}	{[0.7, 0.9], [0.7, 0.8], [0.6, 0.7], [0.5, 0.6]}	{[0.8, 0.9], [0.5, 0.6]}	{[0.3, 0.4], [0.6, 0.7]}
Y_2	{[0.5, 0.7], [0.5, 0.6]}	{[0.7, 0.9], [0.5, 0.6], [0.4, 0.5]}	{[0.5, 0.8]}	{[0.3, 0.5], [0.6, 0.7], [0.8, 0.9]}
Y_3	{[0.5, 0.7], [0.4, 0.6], [0.3, 0.4]}	{[0.8, 0.9], [0.6, 0.7]}	{[0.6, 0.7], [0.5, 0.7], [0.4, 0.5], [0.3, 0.4]}	{[0.3, 0.4], [0.7, 0.8]}
Y_4	{[0.2, 0.3], [0.4, 0.5]}	{[0.5, 0.7]}	{[0.1, 0.2], [0.3, 0.4], [0.5, 0.7]}	{[0.6, 0.8], [0.4, 0.5]}
Y_5	{[0.7, 0.8]}	{[0.5, 0.6], [0.7, 0.9]}	{[0.3, 0.5]}	{[0.3, 0.4], [0.5, 0.7], [0.8, 0.9]}

Table 3. The normalized interval-valued hesitant fuzzy decision matrix $\tilde{A}$

	G_1	G_2	G_3	G_4
Y_1	{[0.5, 0.7], [0.6, 0.7], [0.7, 0.8]}	{[0.7, 0.9], [0.7, 0.8], [0.6, 0.7], [0.5, 0.6]}	{[0.1, 0.2], [0.4, 0.5]}	{[0.6, 0.7], [0.3, 0.4]}
Y_2	{[0.3, 0.5], [0.4, 0.5]}	{[0.7, 0.9], [0.5, 0.6], [0.4, 0.5]}	{[0.2, 0.5]}	{[0.5, 0.7], [0.3, 0.4], [0.1, 0.2]}
Y_3	{[0.3, 0.5], [0.4, 0.6], [0.6, 0.7]}	{[0.8, 0.9], [0.6, 0.7]}	{[0.3, 0.4], [0.3, 0.5], [0.5, 0.6], [0.6, 0.7]}	{[0.6, 0.7], [0.2, 0.3]}
Y_4	{[0.7, 0.8], [0.5, 0.6]}	{[0.5, 0.7]}	{[0.8, 0.9], [0.6, 0.7], [0.3, 0.5]}	{[0.2, 0.4], [0.5, 0.6]}
Y_5	{[0.2, 0.3]}	{[0.5, 0.6], [0.7, 0.9]}	{[0.5, 0.7]}	{[0.6, 0.7], [0.3, 0.5], [0.1, 0.2]}

Step 1. Among the considered criteria, G_j ($j=1,3,4$) are of the cost type and G_2 is of the benefit type; thus, $\tilde{R}=\left(\tilde{r}_{ij}\right)_{5\times 4}$ needs to be transformed into a normalized interval-valued hesitant fuzzy decision matrix $\tilde{A}=\left(\tilde{a}_{ij}\right)_{5\times 4}$ (see Table 3) according to Eq. (38).

Step 2. Utilize the IVHFHOWA operator (Eq. (18)) (whose associated weight vector is $w=(0.1,0.5,0.3,0.1)^T$ and $\theta=3$) to fuse all of the performance values $\tilde{a}_{ij}$ ($j=1,2,3,4$) in the *i*th line of $\tilde{A}$ and then derive the overall performance value $\tilde{a}_i$ ($i=1,2,3,4,5$) of the alternative Y_i ($i=1,2,3,4,5$):

$$\tilde{a}_1 = \left\{ \begin{array}{l} [0.4283,0.6935],[0.4567,0.7132],[0.5182,0.6156],[0.5443,0.6387],[0.4835,0.6935],[0.5106,0.7132], \\ [0.5687,0.6156],[0.5932,0.6387],[0.5449,0.7485],[0.5702,0.7654],[0.6239,0.6808],[0.6463,0.7011], \\ [0.4283,0.6744],[0.4567,0.6950],[0.5182,0.5932],[0.5443,0.6173],[0.4835,0.6744],[0.5106,0.6950], \\ [0.5687,0.5932],[0.5932,0.6173],[0.5449,0.7320],[0.5702,0.7498],[0.6239,0.6611],[0.6463,0.6823], \\ [0.4150,0.6619],[0.4436,0.6830],[0.5058,0.5787],[0.5323,0.6033],[0.4707,0.6619],[0.4981,0.6830], \\ [0.5571,0.5787],[0.5820,0.6033],[0.5330,0.7212],[0.5587,0.7395],[0.6132,0.6482],[0.6361,0.6700], \\ [0.4036,0.6519],[0.4325,0.6735],[0.4952,0.5672],[0.5220,0.5922],[0.4597,0.6519],[0.4874,0.6735], \\ [0.5471,0.5672],[0.5723,0.5922],[0.5227,0.7126],[0.5487,0.7313],[0.6040,0.6380],[0.6272,0.6602] \end{array} \right\}$$

$$\tilde{a}_2 = \left\{ \begin{array}{l} [0.3980,0.6242],[0.3366,0.5354],[0.2764,0.4820],[0.4463,0.6242],[0.3865,0.5354],[0.3271,0.4820], \\ [0.3729,0.5765],[0.3109,0.4817],[0.2504,0.4256],[0.4219,0.5765],[0.3612,0.4817],[0.3013,0.4256], \\ [0.3624,0.5668],[0.3002,0.4708],[0.2396,0.4142],[0.4116,0.5668],[0.3506,0.4708],[0.2906,0.4142] \end{array} \right\}$$

$$\tilde{a}_3 = \left\{ \begin{array}{l} [0.3963,0.5574],[0.3546,0.5173],[0.3963,0.5838],[0.3546,0.5451],[0.4557,0.6119],[0.4154,0.5748], \\ [0.4884,0.6434],[0.4491,0.6082],[0.4446,0.6054],[0.4040,0.5679],[0.4446,0.6299],[0.4040,0.5939], \\ [0.5020,0.6559],[0.4632,0.6216],[0.5332,0.6848],[0.4956,0.6525],[0.5459,0.6574],[0.5088,0.6232], \\ [0.5459,0.6797],[0.5088,0.6470],[0.5972,0.7031],[0.5626,0.6721],[0.6247,0.7289],[0.5916,0.7000], \\ [0.3650,0.5173],[0.3228,0.4754],[0.3650,0.5451],[0.3228,0.5044],[0.4256,0.5748],[0.3845,0.5356], \\ [0.4590,0.6082],[0.4188,0.5709],[0.4142,0.5679],[0.3729,0.5283],[0.4142,0.5939],[0.3729,0.5558], \\ [0.4729,0.6216],[0.4332,0.5851],[0.5051,0.6525],[0.4664,0.6179],[0.5182,0.6232],[0.4800,0.5867], \\ [0.5182,0.6470],[0.4800,0.6120],[0.5714,0.6721],[0.5355,0.6389],[0.6000,0.7000],[0.5655,0.6688] \end{array} \right\}$$

$$\tilde{a}_4 = \left\{ \begin{array}{l} [0.6709,0.8153],[0.6916,0.8249],[0.5474,0.6879],[0.5731,0.7027],[0.3980,0.5937],[0.4276,0.6114], \\ [0.6527,0.8009],[0.6743,0.8112],[0.5252,0.6661],[0.5516,0.6816],[0.3729,0.5677],[0.4028,0.5861] \end{array} \right\}$$

$$\tilde{a}_5 = \{[0.5043,0.6590],[0.4133,0.6012],[0.3546,0.5245],[0.5270,0.7000],[0.4379,0.6470],[0.3800,0.5757]\}$$

Step 3. According to Definition 2.3, we calculate the scores $s(\tilde{a}_i)$ ($i = 1,2,3,4,5$) of $\tilde{a}_i$ ($i = 1,2,3,4,5$) as follows:

$$s(\tilde{a}_1) = 0.5968,\ s(\tilde{a}_2) = 0.4250,\ s(\tilde{a}_3) = 0.5369,\ s(\tilde{a}_4) = 0.6182,\ s(\tilde{a}_5) = 0.5270$$

Step 4. Because $s(\tilde{a}_4) > s(\tilde{a}_1) > s(\tilde{a}_3) > s(\tilde{a}_5) > s(\tilde{a}_2)$, then we determine the ranking order of alternatives Y_i ($i = 1,2,3,4,5$) as $Y_4 \succ Y_1 \succ Y_3 \succ Y_5 \succ Y_2$. Thus, the best alternative is Y_4.

It is noted that we let $\theta = 3$ in the above analysis. In fact, the parameter θ can be assigned different values based on the decision maker's preferences. To investigate the variation of the ranking of five alternatives regarding the value of the parameter θ, we assign θ the values between 0 and 10, and calculate the scores of these five alternatives. The variations of the scores can be found clearly with respect to the values of the parameter θ in Fig. 1.

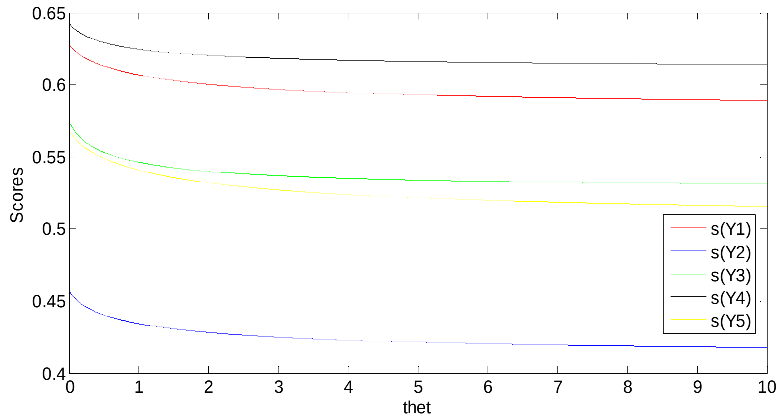

Fig. 1. Scores of alternatives derived by the IVHFHOWA operator

Fig. 1 demonstrates that all of the score functions obtained with the IVHFHOWA operator decrease as θ increases from 0 to 10. Based on this information, we can find that when $\theta \in (0,10]$, the ranking of the four alternatives is $Y_4 \succ Y_1 \succ Y_3 \succ Y_5 \succ Y_2$, and the best choice is Y_4.

If the IVHFHOWA operator in the above example is replaced by the IVHFHOWG operator, then the score functions of five alternatives are shown in Fig. 2. From Fig. 2, we can see that all of the score functions obtained by the IVHFHOWG operator increase as the parameter θ increases from 0 to 10. From Fig. 2, we can also see that when $\theta \in (0,10]$, the ranking of the four alternatives is $Y_4 \succ Y_1 \succ Y_3 \succ Y_5 \succ Y_2$, and the best choice is Y_4.

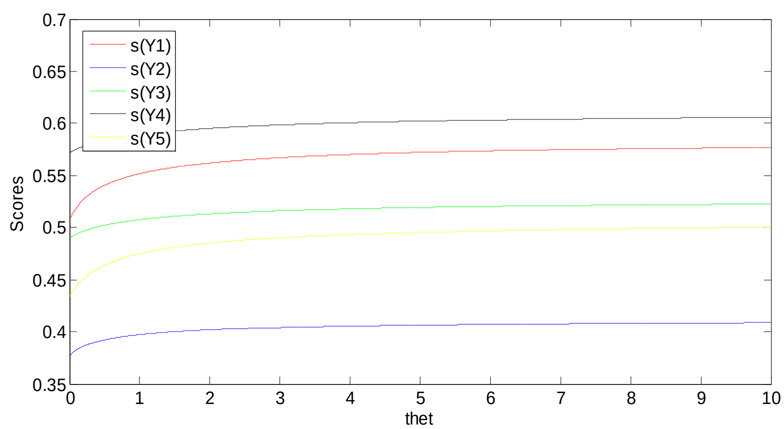

Fig. 2. Scores of alternatives derived by the IVHFHOWG operator

Fig. 3 illustrates the deviation degrees between the scores derived by the IVHFHOWA operator and those derived by the IVHFHOWG operator. From this result, we can find that the values obtained with the IVHFHOWA operator are greater than those obtained with the IVHFHOWG operator, and the deviation values decrease as the value of the parameter θ increases.

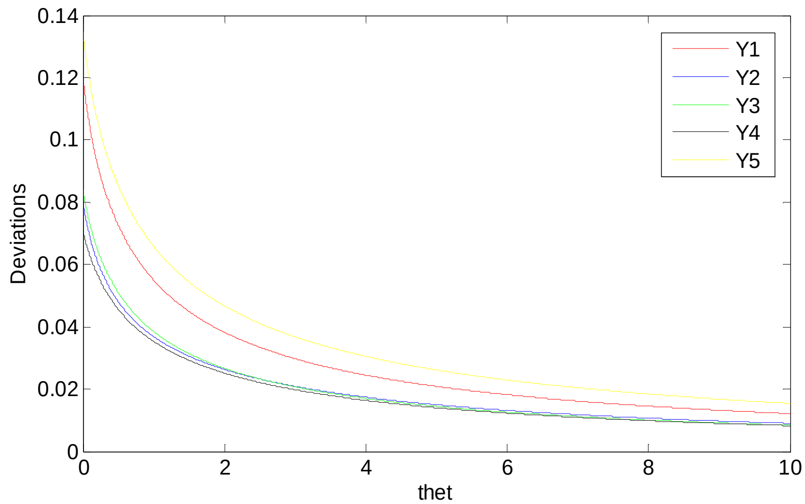

Fig. 3. Deviations of scores derived by the IVHFHOWA and IVHFHOWG operators

Fig. 3 indicates that the IVHFHOWA operator can obtain more favorable (or optimistic) expectations, and can therefore be considered an optimistic operator, while the IVHFHOWG operator has more unfavorable (or pessimistic) expectations, and can therefore be considered a pessimistic operator. The values of the parameter θ can be treated as the optimistic or pessimistic levels. According to Figs. 1, 2, and 3, we can conclude that the decision makers who have a negative perception of the prospects could use the IVHFHOWG operator and choose a smaller value for the parameter θ, while the decision makers who are optimistic could use the IVHFHOWA operator and choose a smaller value for the parameter θ.

If the IVHFFOWA (or IVHFFOWG) operator is replaced by the IVHFHOWA (or IVHFHOWG) operator, then the score functions of alternatives are given in Figs. 4 and 5, respectively. Fig. 4 shows that all of the score functions obtained with the IVHFFOWA operator decrease as the parameter θ increases from 0 to 10, from which we can obtain that when $\theta \in (0,10]$, the ranking of the five alternatives is $Y_4 \succ Y_1 \succ Y_3 \succ Y_5 \succ Y_2$, and the best choice is Y_4.

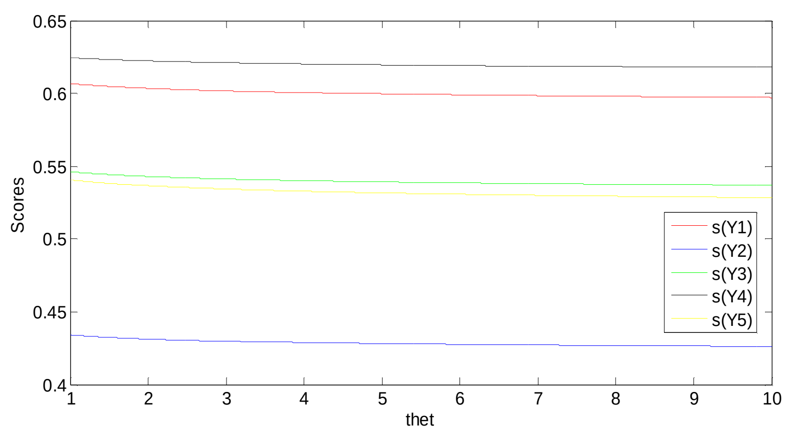

Fig. 4. Scores of alternatives derived by the IVHFFOWA operator

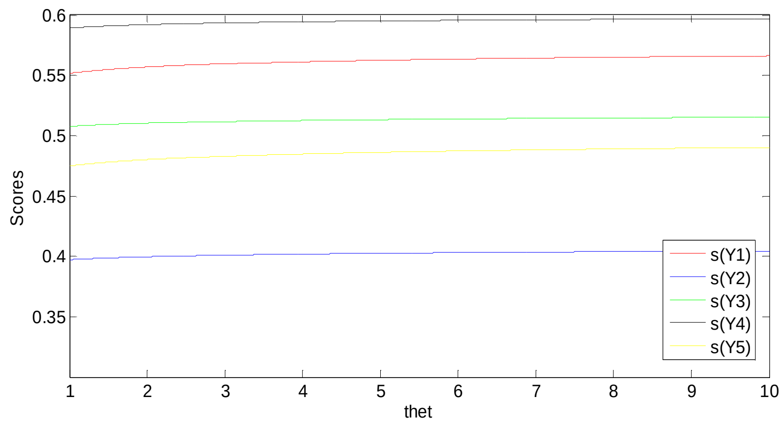

Fig. 5. Scores of alternatives derived by the IVHFFOWG operator

Fig. 5 illustrates that all of the score functions obtained with the IVHFFOWG operator increase as the parameter θ increases from 0 to 10. From this result, we can see that when $\theta \in (0,10]$, the ranking of the four alternatives is $Y_4 \succ Y_1 \succ Y_3 \succ Y_5 \succ Y_2$, and the best choice is Y_4.

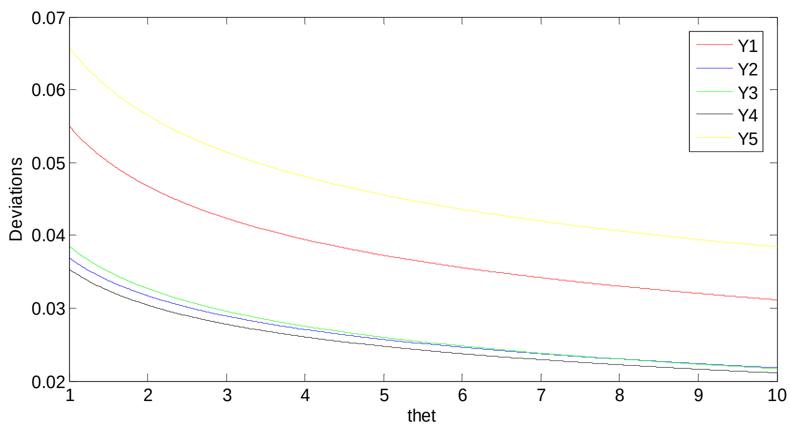

Fig. 6. Deviations of scores obtained by the IVHFFOWA and IVHFFOWG operators

Fig. 6 illustrates the deviation degrees between the score functions derived by the IVHFFOWA operator and those derived by the IVHFFOWG operator. From these results, we can determine that the values obtained with the IVHFFOWA operator are greater than those obtained with the IVHFFOWG operator, and the deviation values decrease as the value of the parameter θ increases.

Fig. 6 indicates that the IVHFFOWA operator can obtain more favorable (or optimistic) expectations, and can therefore be considered an optimistic operator, while the IVHFFOWG operator has more unfavorable (or pessimistic) expectations, and can therefore be considered a pessimistic operator. The values of the parameter θ can be treated as the optimistic or pessimistic levels. According to Figs. 4, 5, and 6, we can conclude that the decision makers who have a negative perception of the prospects could use the IVHFFOWG operator and choose a smaller value for the parameter θ, while the decision makers who are optimistic could use the IVHFFOWA operator and choose a smaller value for the parameter θ.

5.2 Comparative analysis with other methods in the literature

Firstly, Chen et al.'s method [20] is utilized to deal with this issue in order to make a comparison with our method. In Ref. [20], Chen et al. proposed the IVHFOWA operator (see Eq. (14)) based on the algebraic operational laws.

Step 1. Use the IVHFOWA operator to fuse all of the preference values $\tilde{a}_{ij}$ ($j=1,2,3,4$) in the *i*th line of $\tilde{A}$ and then derive the overall performance value $\tilde{a}_i$ ($i=1,2,3,4,5$) of the alternative Y_i ($i=1,2,3,4,5$):

$$\tilde{a}_i = \text{IVHFOWA}\left(\tilde{a}_{i1},\tilde{a}_{i2},\tilde{a}_{i3},\tilde{a}_{i4}\right)$$
$$=\left\{\left[1-\prod_{j=1}^{4}\left(1-\tilde{\gamma}_{i\sigma(j)}^{L}\right)^{\omega_j},1-\prod_{i=1}^{n}\left(1-\tilde{\gamma}_{i\sigma(j)}^{U}\right)^{\omega_j}\right]\middle|\tilde{\gamma}_{i\sigma(1)}\in\tilde{a}_{i\sigma(1)},\tilde{\gamma}_{i\sigma(2)}\in\tilde{a}_{i\sigma(2)},\tilde{\gamma}_{i\sigma(3)}\in\tilde{a}_{i\sigma(3)},\tilde{\gamma}_{i\sigma(4)}\in\tilde{a}_{i\sigma(4)}\right\}$$

The values of $\tilde{a}_i$ ($i=1,2,3,4,5$) are not listed here due to the big data set.

Step 2. According to Definition 2.3, we calculate the scores $s\left(\tilde{a}_i\right)$ ($i=1,2,3,4,5$) of $\tilde{a}_i$ ($i=1,2,3,4,5$) as follows:

$$s\left(\tilde{a}_1\right)=\frac{\sum_{\tilde{\gamma}_1\in\tilde{a}_1}\left(\tilde{\gamma}_1^L+\tilde{\gamma}_1^U\right)}{2l_{\tilde{a}_1}}=0.6067,\ s\left(\tilde{a}_2\right)=0.4342,\ s\left(\tilde{a}_3\right)=0.5462,\ s\left(\tilde{a}_4\right)=0.6246,$$
$$s\left(\tilde{a}_5\right)=0.5408$$

Step 3. Because $s\left(\tilde{a}_4\right)>s\left(\tilde{a}_1\right)>s\left(\tilde{a}_3\right)>s\left(\tilde{a}_5\right)>s\left(\tilde{a}_2\right)$, then we determine the ranking order of alternatives Y_i ($i=1,2,3,4,5$) as $Y_4\succ Y_1\succ Y_3\succ Y_5\succ Y_2$, which is the same as that obtained by our method which explains the validity of our method.

Secondly, Wei and Zhao's method [22] is utilized to deal with this issue in order to make a comparison with our method. In Ref. [22], Wei and Zhao proposed the IVHFEOWA operator (see Eq. (16)) based on the Einstein operational laws.

Step 1. Use the IVHFEOWA operator to fuse all of the preference values $\tilde{a}_{ij}$ ($j=1,2,3,4$) in the *i*th line of $\tilde{A}$ and then derive the overall performance value $\tilde{a}_i$ ($i=1,2,3,4,5$) of the alternative Y_i ($i=1,2,3,4,5$).

$$\tilde{a}_i = \text{IVHFEOWA}\left(\tilde{a}_{i1},\tilde{a}_{i2},\tilde{a}_{i3},\tilde{a}_{i4}\right)$$
$$=\left\{\left[\frac{\prod_{j=1}^{4}\left(1+\tilde{\gamma}_{i\sigma(j)}^{L}\right)^{\omega_j}-\prod_{j=1}^{4}\left(1-\tilde{\gamma}_{i\sigma(j)}^{L}\right)^{\omega_j}}{\prod_{j=1}^{4}\left(1+\tilde{\gamma}_{i\sigma(j)}^{L}\right)^{\omega_j}+\prod_{j=1}^{4}\left(1-\tilde{\gamma}_{i\sigma(j)}^{L}\right)^{\omega_j}},\frac{\prod_{j=1}^{4}\left(1+\tilde{\gamma}_{i\sigma(j)}^{U}\right)^{\omega_j}-\prod_{j=1}^{4}\left(1-\tilde{\gamma}_{i\sigma(j)}^{U}\right)^{\omega_j}}{\prod_{j=1}^{4}\left(1+\tilde{\gamma}_{i\sigma(j)}^{U}\right)^{\omega_j}+\prod_{j=1}^{4}\left(1-\tilde{\gamma}_{i\sigma(j)}^{U}\right)^{\omega_j}}\right]\middle|\tilde{\gamma}_{i\sigma(1)}\in\tilde{a}_{i\sigma(1)},\right.$$
$$\left.\tilde{\gamma}_{i\sigma(2)}\in\tilde{a}_{i\sigma(2)},\tilde{\gamma}_{i\sigma(3)}\in\tilde{a}_{i\sigma(3)},\tilde{\gamma}_{i\sigma(4)}\in\tilde{a}_{i\sigma(4)}\right\}$$

The values of $\tilde{a}_i$ ($i = 1, 2, 3, 4, 5$) are not listed here due to the big data set.

Step 2. According to Definition 2.3, we calculate the scores $s(\tilde{a}_i)$ ($i = 1, 2, 3, 4, 5$) of $\tilde{a}_i$ ($i = 1, 2, 3, 4, 5$) as follows:

$$s(\tilde{a}_1) = 0.6003,\ s(\tilde{a}_2) = 0.4282,\ s(\tilde{a}_3) = 0.5399,\ s(\tilde{a}_4) = 0.6203,\ s(\tilde{a}_5) = 0.5320$$

Step 3. Because $s(\tilde{a}_4) > s(\tilde{a}_1) > s(\tilde{a}_3) > s(\tilde{a}_5) > s(\tilde{a}_2)$, then we determine the ranking order of alternatives Y_i ($i = 1, 2, 3, 4, 5$) as $Y_4 \succ Y_1 \succ Y_3 \succ Y_5 \succ Y_2$, which is the same as that obtained by our method which also explains the validity of our method.

Thirdly, for further comparison, the method proposed by Zhang and Wu in [25] is adopted to deal with this issue. In Ref. [25], an A-IVHFWA operator ("non-ordered" operator) was proposed as follows:

$$\text{A-IVHFWA}(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n) = \mathop{\oplus}_{i=1}^{n}(\omega_i \tilde{h}_i) = \left\{ \left[f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\tilde{\gamma}_i^L) \right), f^{-1}\left(\sum_{i=1}^{n} \omega_i f(\tilde{\gamma}_i^U) \right) \right] \middle| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2, \cdots, \tilde{\gamma}_n \in \tilde{h}_n \right\}$$

When $f(t) = \log\left(\frac{1 + (\theta - 1)t}{1 - t} \right)$, $\theta > 0$, the A-IVHFWA operator is reduced to the IVHFHWA operator [33]:

$$\text{IVHFHWA}(\tilde{h}_1, \tilde{h}_2, \cdots, \tilde{h}_n) = \left\{ \left[\begin{array}{c} \dfrac{\prod_{i=1}^{n}(1 + (\theta - 1)\tilde{\gamma}_i^L)^{\omega_i} - \prod_{i=1}^{n}(1 - \tilde{\gamma}_i^L)^{\omega_i}}{\prod_{i=1}^{n}(1 + (\theta - 1)\tilde{\gamma}_i^L)^{\omega_i} + (\theta - 1)\prod_{i=1}^{n}(1 - \tilde{\gamma}_i^L)^{\omega_i}}, \\ \dfrac{\prod_{i=1}^{n}(1 + (\theta - 1)\tilde{\gamma}_i^U)^{\omega_i} - \prod_{i=1}^{n}(1 - \tilde{\gamma}_i^U)^{\omega_i}}{\prod_{i=1}^{n}(1 + (\theta - 1)\tilde{\gamma}_i^U)^{\omega_i} + (\theta - 1)\prod_{i=1}^{n}(1 - \tilde{\gamma}_i^U)^{\omega_i}} \end{array} \right] \middle| \tilde{\gamma}_1 \in \tilde{h}_1, \tilde{\gamma}_2 \in \tilde{h}_2, \cdots, \tilde{\gamma}_n \in \tilde{h}_n \right\} \quad (41)$$

Step 1. Use the IVHFHWA operator (suppose that $\theta = 3$) to fuse all of the preferences $\tilde{a}_{ij}$ ($j = 1, 2, 3, 4$) in the *i*th line of $\tilde{A}$ and then derive the overall performance $\tilde{a}_i$ ($i = 1, 2, 3, 4, 5$) of alternative Y_i ($i = 1, 2, 3, 4, 5$):

Step 2. According to Definition 2.3, we calculate the score values $s(\tilde{a}_i)$ ($i = 1, 2, 3, 4, 5$) of $\tilde{a}_i$ ($i = 1, 2, 3, 4, 5$) as follows:

$$s(\tilde{a}_1) = 0.4961,\ s(\tilde{a}_2) = 0.4482,\ s(\tilde{a}_3) = 0.6182,\ s(\tilde{a}_4) = 0.5881,\ s(\tilde{a}_5) = 0.5852$$

Step 3. Because $s(\tilde{a}_3) > s(\tilde{a}_4) > s(\tilde{a}_5) > s(\tilde{a}_1) > s(\tilde{a}_2)$, then we determine the ranking order of alternatives Y_i ($i = 1, 2, 3, 4, 5$) as $Y_3 \succ Y_4 \succ Y_5 \succ Y_1 \succ Y_2$, which is slightly different from the results derived by our approach and also Chen et al.'s method and Wei and Zhao's method.

For better comparison, the ranking orders derived by four different methods are summarized in Table 4:

Table 4. The ranking orders derived by four different methods

Methods	Aggregation operators	Ranking orders
Our method	A-IVHFOWA	$Y_4 \succ Y_1 \succ Y_3 \succ Y_5 \succ Y_2$
Chen et al.'s method [20]	IVHFOWA	$Y_4 \succ Y_1 \succ Y_3 \succ Y_5 \succ Y_2$
Wei and Zhao's method [22]	IVHFEOWA	$Y_4 \succ Y_1 \succ Y_3 \succ Y_5 \succ Y_2$
Zhang and Wu's method [25]	A-IVHFWA	$Y_3 \succ Y_4 \succ Y_5 \succ Y_1 \succ Y_2$

Table 4 shows that both Approach I, Chen et al.'s method, and Wei and Zhao's method produce the same ranking, where the optimal alternative is Y_4, while Zhang and Wu's method produces the different ranking, where the optimal alternative is Y_3. That is because the former three methods adopt "ordered" weighted operators (the IVHFHOWA, IVHFOWA and IVHFEOWA operators), while the latter method adopts "non-ordered" weighted operator (the IVHFHWA operator).

1) Comparison among our method, Chen et al.'s method [20], and Wei and Zhao's method [22]

Chen et al.'s method [20] adopts the IVHFOWA operator, Wei and Zhao's method [22] adopts the IVHFEOWA operator, and our method adopts the A-IVHFOWA operator. It is well known that the IVHFOWA and IVHFEOWA operators are two special cases of the A-IVHFOWA operator when the additive generator $g(t) = -\log(t)$ and $g(t) = \log\left(\frac{2-t}{t}\right)$, respectively. When we assign different forms to the additive generator g, we can obtain several special interval-valued hesitant fuzzy aggregation operators. Therefore, the proposed operators and method in this paper can provide us more choices and flexibility than Chen et al.'s and Wei and Zhao's operators and methods.

2) Comparison between our method and Zhang and Wu's method [25]

Compared with Zhang and Wu's method [25], our method assigns the largest and smallest arguments smaller weights, which can relieve the influence of "unfair" arguments on the decision results. For example, in Zhang and Wu's method, the smallest IVHFE $\tilde{a}_{13} = \{[0.1, 0.2], [0.4, 0.5]\}$ in the first line of $\tilde{A} = (\tilde{a}_{ij})_{m \times n}$ is assigned a largest weight ($\omega(\tilde{a}_{13}) = 0.5$), while in our method, $\tilde{a}_{13}$ is assigned a smallest weight ($w(\tilde{a}_{13}) = 0.1$) in order to relieve its influence on the decision result; therefore, a different optimal alternative Y_4 is obtained. Therefore, our method makes the decision making more reasonable and reliable than Zhang and Wu's method.

From the above analysis, we can conclude that our method is more flexible, reasonable, and reliable than Chen et al.'s method [20], Wei and Zhao's method [22], and Zhang and Wu's method [25].

6 Conclusions

In this paper, some new ordered weighted aggregation operators for IVHFEs based on Archimedean t-norm and t-conorm, such as the A-IVHFOWA operator and A-IVHFOWG operators, are proposed, and various properties of these operators are investigated. Then, they are applied to establish a method for solving the MCDM problems in which the criterion values are given in the form of IVHFEs. We have proved that the proposed operators are a generalization of the existing operators based on algebraic t-norm and t-conorm, Einstein t-norm and t-conorm, and Hamacher t-norm and t-conorm, and thus they are more general and more flexible. In addition, this paper has made some comparisons of the proposed method and the previous work and analyzed their differences in details through a numerical example. In further research, it is necessary and meaningful to give the applications of the developed operators to the other domains such as pattern recognition, fuzzy cluster analysis, and uncertain programming.

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Competing Interests

Author has declared that no competing interests exist.

References

[1] Torra V. Hesitant fuzzy sets. International Journal of Intelligent Systems. 2010;25:529–539.

[2] Torra V, Narukawa Y. On hesitant fuzzy sets and decision, in: The 18th IEEE International Conference on Fuzzy Systems, Jeju Island, Korea. 2009;1378–1382.

[3] Zadeh LA. Fuzzy sets. Information and Control. 1965;8:338–353.

[4] Atanassov K. Intuitionistic fuzzy sets. Fuzzy Sets and Systems. 1986;20:87–96.

[5] Zadeh LA. The concept of a linguistic variable and its application to approximate reasoning, Part 1. Information Sciences. 1975;8:199–249.

[6] Atanassov K, Gargov G. Interval valued intuitionistic fuzzy sets. Fuzzy Sets and Systems. 1989;31:343–349.

[7] Dubois D, Prade H. Fuzzy sets and systems: Theory and applications. New York: Academic Press; 1980.

[8] Miyamoto S. Multisets and fuzzy multisets. In Z.-Q. Liu & S. Miyamoto (Eds.), Soft Computing and Human-Centered Machines. Berlin: Springer. 2000;9–33.

[9] Miyamoto S. Remarks on basics of fuzzy sets and fuzzy multisets. Fuzzy Sets and Systems. 2005;156:427–431.

[10] Wei GW. Hesitant fuzzy prioritized operators and their application to multiple attribute decision making. Knowledge-Based Systems. 2012;31:176–182.

[11] Xia MM, Xu ZS. Hesitant fuzzy information aggregation in decision making. International Journal of Approximate Reasoning. 2011;52:395–407.

[12] Xia MM, Xu ZS, Chen N. Some hesitant fuzzy aggregation operators with their application in group decision making. Group Decision Negotiation. 2013;22:259–279.

[13] Xu ZS, Xia MM. Distance and similarity measures for hesitant fuzzy sets. Information Sciences. 2011a;181:2128–2138.

[14] Xu ZS, Xia MM. On distance and correlation measures of hesitant fuzzy information. International Journal of Intelligent Systems. 2011b;26:410–425.

[15] Yu DJ. Some hesitant fuzzy information aggregation operators based on Einstein operational laws. International Journal of Intelligent Systems. 2014;29(4):320–340.

[16] Zhou LY, Zhao XF, Wei GW. Hesitant fuzzy Hamacher aggregation operators and their application to multiple attribute decision making. Journal of Intelligent and Fuzzy Systems. 2014;26(6):2689–2699.

[17] Zhu B, Xu ZS. Hesitant fuzzy Bonferroni means for multi-criteria decision making. Journal of the Operational Research Society. 2013;64:1831–1840.

[18] Zhu B, Xu ZS, Xia MM. Hesitant fuzzy geometric Bonferroni means. Information Sciences. 2012; 205:72–85.

[19] Chen N, Xu ZS, Xia MM. Correlation coefficients of hesitant fuzzy sets and their applications to clustering analysis. Applied Mathematical Modelling. 2013a;37(4):2197–2211.

[20] Chen N, Xu ZS, Xia MM. Interval-valued hesitant preference relations and their applications to group decision making. Knowledge-Based Systems. 2013b;37:528–540.

[21] Zhang ZM, Wang C, Tian DZ, Li K. Induced generalized hesitant fuzzy operators and their application to multiple attribute group decision making. Computers & Industrial Engineering. 2014; 67:116–138.

[22] Wei GW, Zhao XF. Induced hesitant interval-valued fuzzy Einstein aggregation operators and their application to multiple attribute decision making. Journal of Intelligent & Fuzzy Systems. 2013;24(4): 789–803.

[23] Klir G, Yuan B. Fuzzy sets and fuzzy logic: Theory and applications. Prentice Hall, Upper Saddle River, N. J; 1995.

[24] Nguyen HT, Walker EA. A first course in fuzzy logic. CRC Press, Boca Raton, Florida; 1997.

[25] Zhang ZM, Wu C. Some interval-valued hesitant fuzzy aggregation operators based on Archimedean t-norm and t-conorm with their application in multi-criteria decision making. Journal of Intelligent and Fuzzy Systems, in press.

[26] Yager RR. On ordered weighted averaging aggregation operators in multicriteria decision making. IEEE Trans. Fuzzy Syst. Man Cybern. 1988;18:183–190.

[27] Xu ZS, Da QL. The uncertain OWA operator. International Journal of Intelligent Systems. 2002;17:569–575.

[28] Klement EP, Mesiar R (Eds.). Logical, algebraic, analytic, and probabilistic aspects of triangular norms. Elsevier, New York; 2005.

[29] Ahn BS. The uncertain OWA aggregation with weighting functions having a constant level of orness. International Journal of Intelligent Systems. 2006;21:469–483.

[30] Beliakov G. Learning weights in the generalized OWA operators. Fuzzy Optimization and Decision Making. 2005;4:119–130.

[31] Xu ZS. Intuitionistic fuzzy aggregation operators. IEEE Transactions on Fuzzy Systems. 2007;15:1179–1187.

[32] Xu ZS, Da QL. The ordered weighted geometric averaging operator. International Journal of Intelligent Systems. 2002;17:709–716.

[33] Xu ZS, Yager RR. Some geometric aggregation operators based on intuitionistic fuzzy sets. Int. J. Gen. Syst. 2006;35:417–433.

[34] Yager RR. Generalized OWA aggregation operators. Fuzzy Optimization and Decision Making. 2004;3:93–107.

[35] Zhou LG, Chen HY, Merigó JM, Gil-Lafuente AM. Uncertain generalized aggregation operators. Expert Syst. Appl. 2012;39:1105–1117.

[36] Xu ZS, Hu H. Projection models for intuitionistic fuzzy multiple attribute decision making. International Journal of Information Technology and Decision Making. 2010;9:267–280.

[37] Herrera F, Herrera-Viedma E. Linguistic decision analysis: Steps for solving decision problems under linguistic information. Fuzzy Sets and Systems. 2000;115:67–82.

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