Asian Research Journal of Mathematics

8(2): 1-12, 2018; Article no.ARJOM.38369
ISSN: 2456-477X

Exponentiated Transmuted Generalized Inverse Weibull Distribution a Generalization of the Generalized Inverse Weibull Distribution

Yassmen, Y. Abdelall[1*]

[1]Department of Mathematical Statistics, Institute of Statistical Studies and Research, Cairo University, Egypt.

Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/ARJOM/2018/38369
Editor(s):
(1) Nikolaos Dimitriou Bagis, Department of Informatics and Mathematics, Aristotelian University of Thessaloniki, Greece.
Reviewers:
(1) Thomas L. Toulias, Technological Educational Institute of Athens, Greece.
(2) Marwa Osman Mohamed, Zagazig University, Egypt.
Complete Peer review History: http://www.sciencedomain.org/review-history/22806

Received: 23rd November 2017
Accepted: 19th December 2017
Published: 23rd January 2018

Original Research Article

Abstract

This paper introduces a new generalization of the generalized inverse Weibull distribution. This new distribution is named exponentiated transmuted generalized inverse Weibull distribution, which contains a number of distributions as special cases. The properties of the new distribution are discussed and explicit expressions for the quantiles, moments, moment generating function and order statistics are derived. Estimation of the model parameters is performed by maximum likelihood method. Finally, the usefulness of the distribution for modeling data is illustrated using real data.

Keywords: Exponentiated transmuted generalized inverse Weibull distribution; quantiles; moment generating function; order statistics; maximum likelihood estimation.

1 Introduction

The statistics literature is filled with hundreds of lifetime distributions for describing and predicting real world phenomena. These distributions have been extensively used for modelling and analysis of lifetime

**Corresponding author: E-mail: yassmena_youssef@yahoo.com, yassmenamalek@gmail.com;*

data in different areas of research, like engineering, medicine, reliability, etc. In this regard, it is observed that the inverse Weibull distribution is extensively used as it is found to provide reasonable fit in many practical situations.

The transmuted generalized inverse Weibull (TGIW) distribution has been introduced by Merovci et al. [1]. The cumulative distribution function (cdf) of the TGIW is defined as

$$G_{\text{TGIW}}(t) = e^{-\gamma(\alpha t)^{-\beta}}\left[1+\lambda-\lambda\, e^{\frac{-\gamma}{(\alpha t)^{\beta}}}\right],$$

where α is a scale parameter, β, γ are shape parameters, and λ is the transmuting parameter. Many authors dealing with the generalization of some well-known distributions. Aryal and Tsokos [2] defined the transmuted generalized extreme value distribution and they studied transmuted gumbel distribution. Aryal and Tsokos [3] presented the transmuted Weibull distribution. Also Aryal [4] studied the various structural properties of transmuted log-logistic distribution. Khan and King [5] investigated the transmuted modified Weibull distribution. Also Elbatal and Muhammed [6] studied the exponentiated generalized inverse Weibull distribution. In this paper, we introduce and study several mathematical properties of a new generalization model of the transmuted generalized inverse Weibull distribution called the exponentiated transmuted generalized inverse Weibull (ETGIW) distribution by introducing another shape parameter.

The paper is organized as follows: In Section 2, we introduce the ETGIW distribution and some special sub-models are derived. Section 3 discusses some important statistical properties including quantile, moments and moment generating function are studied. Some distributions of order statistics models are expressed in Section 4. Estimation of the parameters by maximum likelihood method is presented in Section 5. The usefulness of the distribution for modeling real life data is illustrated in section 6. Finally, we make some concluding remarks on our study.

2 Exponentiated Transmuted Generalized Inverse Weibull Distribution

The five parameter exponentiated transmuted generalized inverse Weibull distribution ETGIW ($\alpha, \beta, \gamma, \nu$) is given by the cdf

$$F(t) = \left[e^{-\gamma(\alpha t)^{-\beta}}\left[1+\lambda-\lambda\, e^{-\gamma(\alpha t)^{-\beta}}\right]\right]^{\nu}, \nu = 1,2,3,\dots \quad, \tag{1}$$

where α is a scale parameter, β, γ, ν are shape parameters, and λ is the transmuting parameter. Differentiating (1) with respect to t , and doing the necessary simplifications, gives the density function as

$$f(t) = \nu\alpha\gamma\beta\,(\alpha t)^{-\beta-1}\, e^{-\gamma(\alpha t)^{-\beta}}\left[1+\lambda-2\lambda\, e^{-\gamma(\alpha t)^{-\beta}}\right]\times\left[e^{-\gamma(\alpha t)^{-\beta}}\left[1+\lambda-\lambda\, e^{-\gamma(\alpha t)^{-\beta}}\right]\right]^{\nu-1}. \tag{2}$$

The survival and hazard (failure) rate functions of the (ETGIW) distribution are given by:

$$s(t) = 1-\left[e^{-\gamma(\alpha t)^{-\beta}}\left[1+\lambda-\lambda\, e^{-\gamma(\alpha t)^{-\beta}}\right]\right]^{\nu},$$

$$h(t) = \frac{\nu\alpha\gamma\beta\,(\alpha t)^{-\beta-1}\, e^{-\gamma(\alpha t)^{-\beta}}\left[1+\lambda-2\lambda\, e^{-\gamma(\alpha t)^{-\beta}}\right]\times\left[e^{-\gamma(\alpha t)^{-\beta}}\left[1+\lambda-\lambda\, e^{-\gamma(\alpha t)^{-\beta}}\right]\right]^{\nu-1}}{1-\left[e^{-\gamma(\alpha t)^{-\beta}}\left[1+\lambda-\lambda\, e^{-\gamma(\alpha t)^{-\beta}}\right]\right]^{\nu}}.$$

Some plots of the possible shapes of the density function, distribution function and hazard rate function of the (ETGIW) distribution for selected values of parameters.

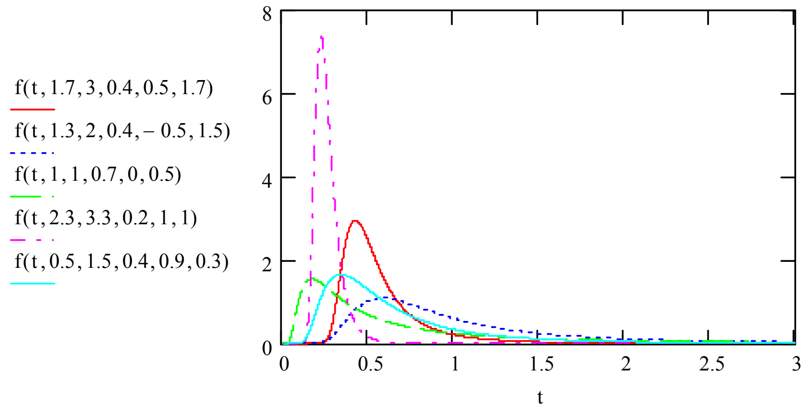

Fig. 1. Exponentiated transmuted generalized inverse Weibull pdf

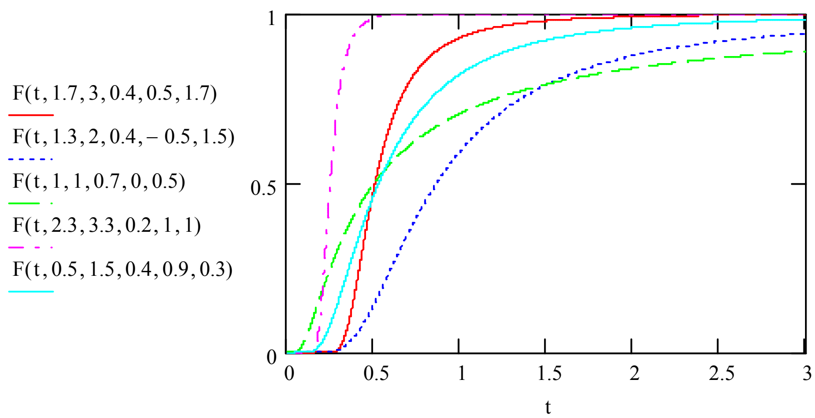

Fig. 2. Exponentiated transmuted generalized inverse Weibull cdf

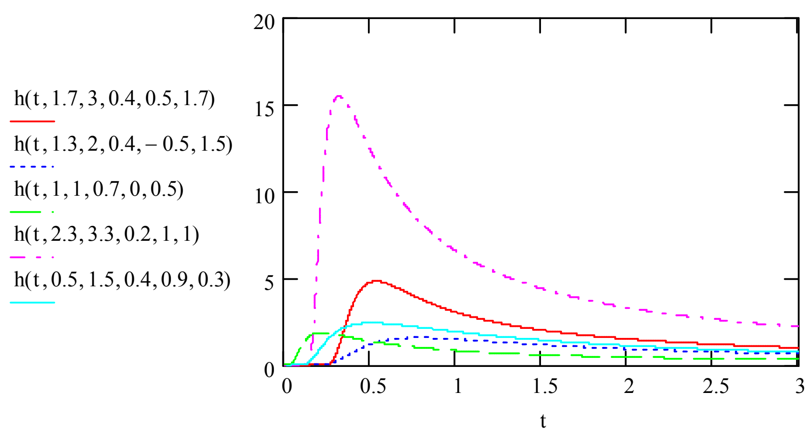

Fig. 3. Exponentiated transmuted generalized inverse Weibull hazard rate function

The exponentiated transmuted generalized inverse Weibull distribution is very flexible (as seen from Table 1). This is so since several other distributions follow as special cases from the (ETGIW) distribution, by selecting the appropriate values of the parameters as shown below:

Table 1. The ETGIW distribution submodels

Parameters	Submodels
$\lambda = 0$	Exponentiated Generalized Inverse Weibull
$\gamma = 1$	Exponentiated Transmuted Inverse Weibull
$\gamma = 1$, $\beta = 1$	Exponentiated Transmuted Inverse Exponential
$\gamma = 1$, $\beta = 2$	Exponentiated Transmuted Inverse Rayleigh
$\alpha = 1$	Exponentiated Transmuted Frechet
$\lambda = 0$, $\gamma = 1$	Exponentiated Inverse Weibull
$\lambda = 0$, $\gamma = 1$, $\beta = 1$	Exponentiated Inverse Exponential
$\lambda = 0$, $\gamma = 1$, $\beta = 2$	Exponentiated Inverse Rayleigh
$\alpha = 1$, $\lambda = 0$	Exponentiated Frechet
$\nu = 1$	Transmuted Generalized Inverse Weibull
$\nu = 1$, $\gamma = 1$	Transmuted Inverse Weibull
$\nu = 1$, $\gamma = 1$, $\beta = 1$	Transmuted Inverse Exponential
$\nu = 1$, $\gamma = 1$, $\beta = 2$	Transmuted Inverse Rayleigh
$\nu = 1$, $\alpha = 1$	Transmuted Frechet
$\nu = 1$, $\lambda = 0$	Generalized Inverse Weibull
$\nu = 1$, $\lambda = 0$, $\gamma = 1$	Inverse Weibull
$\nu = 1$, $\lambda = 0$, $\gamma = 1$, $\beta = 1$	Inverse Exponential
$\nu = 1$, $\lambda = 0$, $\gamma = 1$, $\beta = 2$	Inverse Rayleigh
$\nu = 1$, $\lambda = 0$, $\alpha = 1$	Frechet

Table 1 shows the specific values of the parameters used to generate the above mentioned nineteen special cases.

3 Statistical Properties

This section explains statistical properties of the (ETGIW) distribution including the quantiles, random number generation function, moments and moment generating function.

3.1 Quantiles

The quantile function t_q of the (ETGIW) distribution is the solution of the following equation

$$t_q = \frac{1}{\alpha}\left\{\frac{1}{\gamma}\log\left[\frac{1+\lambda-\lambda e^{-\gamma(\alpha t_q)^{-\beta}}}{q^{\frac{1}{\nu}}}\right]\right\}^{-\frac{1}{\beta}}, \quad 0 \le q \le 1. \tag{3}$$

The above equation has no closed form solution in t_q, so we have to use numerical techniques, such as Newton- Raphson method, to get the quantile. By putting q=0.5 in (3) one gets the median.

3.2 Random number generation

A random variate T from (ETGIW) distribution can be generated as t_u according to (3), where q is replaced by $U \sim U(0,1)$.

3.3 Moments

In this subsection we discuss the r[th] moment for (ETGIW) distribution is given by the following theorem.

Theorm (3.1) If T is a continuous random variable has the ETGIW distribution, then the r[th] non-central moments, $\mu_r^{'} = E(T^r)$ of T is given by the following

$$\mu_r^{'} = \frac{\nu \gamma^{\frac{r}{\beta}}}{\alpha^r} \Gamma\left(1 - \frac{r}{\beta}\right) \sum_{i=0}^{\nu-1} \sum_{j=0}^{i} \binom{\nu-1}{i} \binom{i}{j} \lambda^i (-1)^{i+j} \times \left[(1+\lambda)(\nu+i-j)^{\frac{r}{\beta}-1} - 2\lambda(\nu+i-j+1)^{\frac{r}{\beta}-1}\right]. \tag{4}$$

Proof: Starting with

$$\begin{aligned} \mu_r^{'} &= \int_0^\infty t^r f(t)\, dt \\ &= \int_0^\infty t^r \nu\alpha\gamma\beta (\alpha t)^{-\beta-1} e^{-\gamma(\alpha t)^{-\beta}} \left[1+\lambda - 2\lambda e^{-\gamma(\alpha t)^{-\beta}}\right] \times \left[e^{-\gamma(\alpha t)^{-\beta}} \left[1+\lambda - \lambda e^{-\gamma(\alpha t)^{-\beta}}\right]\right]^{\nu-1} dt \\ &= \nu\gamma\beta\alpha^{-\beta} \int_0^\infty t^{r-\beta-1} e^{-\gamma\nu(\alpha t)^{-\beta}} \left[1+\lambda - 2\lambda e^{-\gamma(\alpha t)^{-\beta}}\right] \times \left[1 - \lambda\left(e^{-\gamma(\alpha t)^{-\beta}} - 1\right)\right]^{\nu-1} dt. \end{aligned} \tag{5}$$

From the fact that $0 \prec \left[1 - \lambda\left(e^{-\gamma(\alpha t)^{-\beta}} - 1\right)\right] \prec 1$, since $|\lambda| \leq 1$, we obtain

$$\left[1 - \lambda\left(e^{-\gamma(\alpha t)^{-\beta}} - 1\right)\right]^{\nu-1} = \sum_{i=0}^{\nu-1} \binom{\nu-1}{i} (-\lambda)^i \left(e^{-\gamma(\alpha t)^{-\beta}} - 1\right)^i. \tag{6}$$

Substituting from (6) into (5), we get

$$\mu_r^{'} = \sum_{i=0}^{\nu-1} \binom{\nu-1}{i} (-\lambda)^i \nu\gamma\beta\alpha^{-\beta} \int_0^\infty t^{r-\beta-1} e^{-\gamma\nu(\alpha t)^{-\beta}} \left(e^{-\gamma(\alpha t)^{-\beta}} - 1\right)^i \left[1+\lambda - 2\lambda e^{-\gamma(\alpha t)^{-\beta}}\right] dt.$$

Using binomial expansion $\left(e^{-\gamma(\alpha t)^{-\beta}} - 1\right)^i$, we obtain

$$\mu_r^{'} = \sum_{i=0}^{\nu-1}\sum_{j=0}^{i}\binom{\nu-1}{i}\binom{i}{j}(-1)^{i+j}\,\lambda^i\,\nu\gamma\beta\alpha^{-\beta}\int_0^\infty t^{r-\beta-1}\,e^{-\gamma(\nu+i-j)(\alpha t)^{-\beta}}\left[1+\lambda-2\lambda\,e^{-\gamma(\alpha t)^{-\beta}}\right]dt$$

$$=(1+\lambda)\sum_{i=0}^{\nu-1}\sum_{j=0}^{i}\binom{\nu-1}{i}\binom{i}{j}(-1)^{i+j}\,\lambda^i\,\nu\gamma\beta\alpha^{-\beta}\int_0^\infty t^{r-\beta-1}\,e^{-\gamma(\nu+i-j)(\alpha t)^{-\beta}}\,dt$$

$$-2\sum_{i=0}^{\nu-1}\sum_{j=0}^{i}\binom{\nu-1}{i}\binom{i}{j}(-1)^{i+j}\,\lambda^{i+1}\,\nu\gamma\beta\alpha^{-\beta}\int_0^\infty t^{r-\beta-1}\,e^{-\gamma(\nu+i-j+1)(\alpha t)^{-\beta}}\,dt\,.$$

Now let $x = x(t) = \gamma(\nu+i-j)(\alpha t)^{-\beta}$. Then , $t = t(x) = \alpha^{-1}\left[\gamma(\nu+i-j)\right]^{1/\beta} x^{-1/\beta}$, and therefore equation 4 is finally derived completing the proof. As a result, the expected value E(T) and the variance Var(T) of the exponentiated transmuted generalized inverse Weibull random variable T are, respectively, given by

$$\mathrm{E}(\mathrm{T}) = \frac{\nu\gamma^{\frac{1}{\beta}}}{\alpha}\Gamma\left(1-\frac{1}{\beta}\right)\sum_{i=0}^{\nu-1}\sum_{j=0}^{i}\binom{\nu-1}{i}\binom{i}{j}\lambda^i(-1)^{i+j}$$
$$\times\left[(1+\lambda)(\nu+i-j)^{\frac{1}{\beta}-1}-2\lambda(\nu+i-j+1)^{\frac{1}{\beta}-1}\right],$$

and the variance is

$$Var[\mathrm{T}] = \mathrm{E}(\mathrm{T}^2) - \mathrm{E}^2(\mathrm{T}),$$

Where $\mathrm{E}(\mathrm{T}^2)$ is given by

$$\mathrm{E}(\mathrm{T}^2) = \frac{\nu\gamma^{\frac{2}{\beta}}}{\alpha^2}\Gamma\left(1-\frac{2}{\beta}\right)\sum_{i=0}^{\nu-1}\sum_{j=0}^{i}\binom{\nu-1}{i}\binom{i}{j}\lambda^i(-1)^{i+j}$$
$$\times\left[(1+\lambda)(\nu+i-j)^{\frac{2}{\beta}-1}-2\lambda(\nu+i-j+1)^{\frac{2}{\beta}-1}\right].$$

The n^{th} central moments, m_n can be obtained easily from the r^{th} non-central moments through the relation

$$\mathrm{m}_\mathrm{n} = \mathrm{E}\left[T-\mu\right]^n = \sum_{r=0}^{n}\binom{n}{r}(-\mu)^{n-r}\,\mathrm{E}(T^r).$$

Thus the n^{th} central moments of the ETGIW distribution is given by

$$m_n = \nu\sum_{r=0}^{n}\sum_{i=0}^{\nu-1}\sum_{j=0}^{i}\binom{\nu-1}{i}\binom{i}{j}\binom{n}{r}\lambda^i(-1)^{i+j}(-\mu)^{n-r}\,\gamma^{\frac{r}{\beta}}\,\alpha^{-r}\,\Gamma\left(1-\frac{r}{\beta}\right)$$
$$\times\left[(1+\lambda)(\nu+i-j)^{\frac{r}{\beta}-1}-2\lambda(\nu+i-j+1)^{\frac{r}{\beta}-1}\right].$$

Based on (4), the coefficient of variation, coefficient of skewness and coefficient of kurtosis of ETGIW distribution can be obtained according to the following relations

$$CV = \frac{\sqrt{m_2}}{m_1} = \sqrt{\frac{\mu_2^{'}}{\mu_1^{'2}} - 1} \ ,$$

$$CS = \frac{m_3}{(m_2)^{3/2}} = \frac{\mu_3^{'} - 3\mu_2^{'}\mu_1^{'} + 2\mu_1^{'3}}{\left(\mu_2^{'} - \mu_1^{'2}\right)^{3/2}} \ ,$$

$$CK = \frac{m_4}{(m_2)^2} = \frac{\mu_4^{'} - 4\mu_3^{'}\mu_1^{'} + 6\mu_2^{'}\mu_1^{'2} - 3\mu_1^{'4}}{\left(\mu_2^{'} - \mu_1^{'2}\right)^2} \ .$$

3.4 Moment generating function

In this subsection we derived the moment generating function of the ETGIW distribution.

Theorem (3.2) If T is a continuous random variable has the ETGIW distribution, then the moment generating function of T is given by

$$M_T(t) = \nu \sum_{r=0}^{\infty}\sum_{i=0}^{\nu-1}\sum_{j=0}^{i} \binom{\nu-1}{i}\binom{i}{j} \lambda^i (-1)^{i+j} \frac{\gamma^{\frac{r}{\beta}}}{r!}\left(\frac{t}{\alpha}\right)^r \Gamma\left(1-\frac{r}{\beta}\right) \times\left[(1+\lambda)(\nu+i-j)^{\frac{r}{\beta}-1} - 2\lambda(\nu+i-j+1)^{\frac{r}{\beta}-1}\right].$$

Proof: The moment generating function of the random variable T is given by

$$M_T(t) = E\left[e^{tT}\right] = \sum_{r=0}^{\infty}\frac{t^r}{r!}\mathrm{E}\left(\mathrm{T}^r\right)$$

$$= \nu \sum_{r=0}^{\infty}\sum_{i=0}^{\nu-1}\sum_{j=0}^{i} \binom{\nu-1}{i}\binom{i}{j} \lambda^i (-1)^{i+j} \frac{\gamma^{\frac{r}{\beta}}}{r!}\left(\frac{t}{\alpha}\right)^r \Gamma\left(1-\frac{r}{\beta}\right) \times\left[(1+\lambda)(\nu+i-j)^{\frac{r}{\beta}-1} - 2\lambda(\nu+i-j+1)^{\frac{r}{\beta}-1}\right].$$

which completes the proof.

4 Order Statistics

In fact, the order statistics have many applications in reliability and life testing. The order statistics play an important role in statistical inference.

4.1 Some distributions of order statistics

Let $T_{(1)} \le T_{(2)} \le \ldots \le T_{(n)}$ be the ordered observations in a random sample of size n drawn from exponentiated transmuted generalized inverse Weibull distribution with cdf F(t) ,given by (1) and pdf f(t) ,given by (2). The pdf of $T_{(j)}$, $j = 1,2,\ldots,n$ is given by

$$f_{T_{(j)}}(t) = \frac{n!}{(j-1)!\,(n-j)!} f(t)\left[F(t)\right]^{j-1}\left[1-F(t)\right]^{n-j}.$$

$$= \frac{n!}{(j-1)!\,(n-j)!} \nu\alpha\gamma\beta\,(\alpha\,t)^{-\beta-1}\,e^{-\gamma(\alpha\,t)^{-\beta}}\left[1+\lambda-2\lambda\,e^{-\gamma(\alpha\,t)^{-\beta}}\right]$$

$$\times\left[e^{-\gamma(\alpha\,t)^{-\beta}}\left[1+\lambda-\lambda\,e^{-\gamma(\alpha\,t)^{-\beta}}\right]\right]^{\nu\,j-1}\times\left\{1-\left[e^{-\gamma(\alpha\,t)^{-\beta}}\left[1+\lambda-\lambda\,e^{-\gamma(\alpha\,t)^{-\beta}}\right]\right]^{\nu}\right\}^{n-j}.$$

Therefore, the pdf of the largest order statistic $T_{(n)}$, the smallest order statistic $T_{(1)}$ and the median order statistic $T_{(m+1)}$ when n=2m+1 (n is odd number) are, respectively, given by

$$f_{T_{(n)}}(t) = n\,\nu\alpha\gamma\beta\,(\alpha\,t)^{-\beta-1}\,e^{-\gamma(\alpha\,t)^{-\beta}}\left[1+\lambda-2\lambda\,e^{-\gamma(\alpha\,t)^{-\beta}}\right]\times\left[e^{-\gamma(\alpha\,t)^{-\beta}}\left[1+\lambda-\lambda\,e^{-\gamma(\alpha\,t)^{-\beta}}\right]\right]^{n\,\nu-1},$$

$$f_{T_{(1)}}(t) = n\,\nu\alpha\gamma\beta\,(\alpha\,t)^{-\beta-1}\,e^{-\gamma(\alpha\,t)^{-\beta}}\left[1+\lambda-2\lambda\,e^{-\gamma(\alpha\,t)^{-\beta}}\right]$$

$$\times\left[e^{-\gamma(\alpha\,t)^{-\beta}}\left[1+\lambda-\lambda\,e^{-\gamma(\alpha\,t)^{-\beta}}\right]\right]^{\nu-1}\times\left\{1-\left[e^{-\gamma(\alpha\,t)^{-\beta}}\left[1+\lambda-\lambda\,e^{-\gamma(\alpha\,t)^{-\beta}}\right]\right]^{\nu}\right\}^{n-1},$$

and

$$f_{T_{(m+1)}} = \frac{(2m+1)!}{((m)!)^2}\nu\alpha\gamma\beta\,(\alpha\,t)^{-\beta-1}\,e^{-\gamma(\alpha\,t)^{-\beta}}\left[1+\lambda-2\lambda\,e^{-\gamma(\alpha\,t)^{-\beta}}\right]$$

$$\times\left[e^{-\gamma(\alpha\,t)^{-\beta}}\left[1+\lambda-\lambda\,e^{-\gamma(\alpha\,t)^{-\beta}}\right]\right]^{\nu\,(m+1)-1}\times\left\{1-\left[e^{-\gamma(\alpha\,t)^{-\beta}}\left[1+\lambda-\lambda\,e^{-\gamma(\alpha\,t)^{-\beta}}\right]\right]^{\nu}\right\}^{m}.$$

4.2 Joint distributions of ith and jth order statistics

The joint pdf of two order statistics $\left(T_{(i)}, T_{(j)}\right)$, for $i, j = 1,\ldots, n,$ of the ETGIW distribution is

$$f_{(T_{(i)},T_{(j)})}(t_i, t_j) = C\left[F(t_i)\right]^{i-1}\left[F(t_j)-F(t_i)\right]^{j-i-1}\left[1-F(t_j)\right]^{n-j} f(t_i)\,f(t_j),$$

$$= C\left[h_i\left\{1+\lambda-\lambda\,h_i\right\}\right]^{\nu\,i-\nu}\times\left[\left(h_j\left\{1+\lambda-\lambda\,h_j\right\}\right)^{\nu}-\left(h_i\left\{1+\lambda-\lambda\,h_i\right\}\right)^{\nu}\right]^{j-i-i}$$

$$\times\left[1-\left(h_j\left\{1+\lambda-\lambda\,h_j\right\}\right)^{\nu}\right]^{n-j}$$

$$\times\,\nu\alpha\gamma\beta\,(\alpha\,t_i)^{-\beta-1}\,h_i\left[1+\lambda-2\lambda\,h_i\right]\times\left[h_i\left[1+\lambda-\lambda\,h_i\right]\right]^{\nu-1}$$

$$\times\,\nu\alpha\gamma\beta\,(\alpha\,t_j)^{-\beta-1}\,h_j\left[1+\lambda-2\lambda\,h_j\right]\times\left[h_j\left[1+\lambda-\lambda\,h_j\right]\right]^{\nu-1}.$$

Where

$$C = \frac{n!}{(i-1)!(j-i-1)!\,(n-j)!} \text{, and } h_i = e^{-\gamma(\alpha t_i)^{-\beta}}.$$

For the special case $i = 1$ and $j = n$ we get the joint distribution of the minimum and maximum as

$$\begin{aligned} f_{(T_{(1)},T_{(n)})}(t_1, t_n) &= n(n-1)[F(t_n) - F(t_1)]^{n-2} f(t_1) f(t_n), \\ &= n(n-1)\left[(h_n\{1+\lambda-\lambda h_n\})^{\nu} - (h_1\{1+\lambda-\lambda h_1\})^{\nu}\right]^{n-2} \\ &\quad \times \nu\alpha\gamma\beta(\alpha t_1)^{-\beta-1} h_1[1+\lambda-2\lambda h_1] \times [h_1[1+\lambda-\lambda h_1]]^{\nu-1} \\ &\quad \times \nu\alpha\gamma\beta(\alpha t_n)^{-\beta-1} h_n[1+\lambda-2\lambda h_n] \times [h_n[1+\lambda-\lambda h_n]]^{\nu-1}. \end{aligned}$$

Note that for $\nu = 1$ yields the order statistics of the transmuted generalized inverse weibull distribution.

5 Estimation of the Parameters

Now we derive the maximum likelihood estimators (MLEs) and discuss inference for the parameters of the ETGIW distribution. Let $(T_1, T_2, ..., T_n)$ be a random sample of size n from an ETGIW$(\nu, \alpha, \beta, \gamma, \lambda)$ distribution then the likelihood function can be written as

$$\begin{aligned} l = \left(\frac{\nu\gamma\beta}{\alpha^{\beta}}\right)^{n} \prod_{i=1}^{n} (t_i)^{-\beta-1} e^{-\gamma\sum_{i=1}^{n}(\alpha t_i)^{-\beta}} \prod_{i=1}^{n}\left[1+\lambda-2\lambda e^{-\gamma(\alpha t_i)^{-\beta}}\right] \\ \times \prod_{i=1}^{n}\left[e^{-\gamma(\alpha t)^{-\beta}}\left[1+\lambda-\lambda e^{-\gamma(\alpha t)^{-\beta}}\right]\right]^{\nu-1}. \end{aligned}$$

Then, the log-likelihood function, ℓ , becomes:

$$\begin{aligned} \ell = n\ln\nu + n\ln\gamma + n\ln\beta - n\beta\ln\alpha - (\beta+1)\sum_{i=1}^{n}\ln(t_i) - \gamma\sum_{i=1}^{n}(\alpha t_i)^{-\beta} \\ + \sum_{i=1}^{n}\ln\left[1+\lambda-2\lambda e^{-\gamma(\alpha t_i)^{-\beta}}\right] + (\nu-1)\sum_{i=1}^{n}\ln\left[e^{-\gamma(\alpha t_i)^{-\beta}}\left[1+\lambda-\lambda e^{-\gamma(\alpha t_i)^{-\beta}}\right]\right]. \end{aligned} \tag{7}$$

The log-likelihood function can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (7). The components of the score vector are given by

$$\frac{\partial\ell}{\partial\nu} = \frac{n}{\nu} + \sum_{i=1}^{n}\ln\left[e^{-\gamma(\alpha t_i)^{-\beta}}\left[1+\lambda-\lambda e^{-\gamma(\alpha t_i)^{-\beta}}\right]\right], \tag{8}$$

$$\begin{aligned} \frac{\partial\ell}{\partial\alpha} = -\frac{n\beta}{\alpha} + \gamma\beta\sum_{i=1}^{n} t_i(\alpha t_i)^{-\beta-1} - \sum_{i=1}^{n}\frac{\left[2\lambda\gamma\beta t_i(\alpha t_i)^{-\beta-1} e^{-\gamma(\alpha t_i)^{-\beta}}\right]}{\left[1+\lambda-2\lambda e^{-\gamma(\alpha t_i)^{-\beta}}\right]} \\ + (\nu-1)\sum_{i=1}^{n}\left[\gamma\beta t_i(\alpha t_i)^{-\beta-1}\left[1-\frac{\lambda e^{-\gamma(\alpha t_i)^{-\beta}}}{\left[1+\lambda-\lambda e^{-\gamma(\alpha t_i)^{-\beta}}\right]}\right]\right], \end{aligned} \tag{9}$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} - n \ln \alpha - \sum_{i=1}^{n} \ln t_i + \gamma \sum_{i=1}^{n} (\alpha\, t_i)^{-\beta} \ln(\alpha\, t_i) - \sum_{i=1}^{n} \frac{\left[2 \lambda \gamma (\alpha\, t_i)^{-\beta} \ln(\alpha\, t_i) e^{-\gamma(\alpha\, t_i)^{-\beta}}\right]}{\left[1 + \lambda - 2\lambda\, e^{-\gamma(\alpha\, t_i)^{-\beta}}\right]}$$
$$+ (\nu - 1) \sum_{i=1}^{n} \left[\gamma (\alpha\, t_i)^{-\beta} \ln(\alpha\, t_i) \left[1 - \frac{\lambda\, e^{-\gamma(\alpha\, t_i)^{-\beta}}}{\left[1 + \lambda - \lambda\, e^{-\gamma(\alpha\, t_i)^{-\beta}}\right]} \right] \right], \tag{10}$$

$$\frac{\partial \ell}{\partial \gamma} = \frac{n}{\gamma} - \sum_{i=1}^{n} (\alpha\, t_i)^{-\beta} + \sum_{i=1}^{n} \frac{\left[2 \lambda (\alpha\, t_i)^{-\beta} e^{-\gamma(\alpha\, t_i)^{-\beta}}\right]}{\left[1 + \lambda - 2\lambda\, e^{-\gamma(\alpha\, t_i)^{-\beta}}\right]}$$
$$- (\nu - 1) \sum_{i=1}^{n} \left[(\alpha\, t_i)^{-\beta} \left[1 - \frac{e^{-\gamma(\alpha\, t_i)^{-\beta}}}{\left[1 + \lambda - \lambda\, e^{-\gamma(\alpha\, t_i)^{-\beta}}\right]} \right] \right], \tag{11}$$

and

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^{n} \frac{\left[1 - 2 e^{-\gamma(\alpha\, t_i)^{-\beta}}\right]}{\left[1 + \lambda - 2\lambda\, e^{-\gamma(\alpha\, t_i)^{-\beta}}\right]} + (\nu - 1) \sum_{i=1}^{n} \left[\frac{\left(1 - e^{-\gamma(\alpha\, t_i)^{-\beta}}\right)}{\left[1 + \lambda - \lambda\, e^{-\gamma(\alpha\, t_i)^{-\beta}}\right]} \right]. \tag{12}$$

We can find the estimates of the unknown parameters by maximum likelihood method by these above non-linear equations to zero and solve them simultaneously. From equation (8), we obtain the maximum likelihood estimate of ν in a closed form as follows

$$\hat{\nu} = \frac{-n}{\sum_{i=1}^{n} \ln \left[e^{-\hat{\gamma}(\hat{\alpha}\, t_i)^{-\hat{\beta}}} \left[1 + \hat{\lambda} - \hat{\lambda}\, e^{-\hat{\gamma}(\hat{\alpha}\, t_i)^{-\hat{\beta}}}\right]\right]}, \tag{13}$$

When estimates $\left(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\lambda}\right)$ known. Substituting from (13) into (9), (10), (11), and (12) , we get the MLEs $\left(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\lambda}\right)$. These equations cannot be solved analytically and statistical software can be used to solve them simultaneously. Therefore, we have to use mathematical package to get the MLE of the unknown parameters. For the five parameters exponentiated transmuted generalized inverse Weibull distribution pdf, all the second order derivatives exist. Thus we have the inverse dispersion matrix is given by

$$\begin{pmatrix} \hat{\nu} \\ \hat{\alpha} \\ \hat{\beta} \\ \hat{\gamma} \\ \hat{\lambda} \end{pmatrix} \sim N \left[\begin{pmatrix} \nu \\ \alpha \\ \beta \\ \gamma \\ \lambda \end{pmatrix}, \begin{pmatrix} \hat{V}_{\nu\nu} & \hat{V}_{\nu\alpha} & \hat{V}_{\nu\beta} & \hat{V}_{\nu\gamma} & \hat{V}_{\nu\lambda} \\ \hat{V}_{\alpha\nu} & \hat{V}_{\alpha\alpha} & \hat{V}_{\alpha\beta} & \hat{V}_{\alpha\gamma} & \hat{V}_{\alpha\lambda} \\ \hat{V}_{\beta\nu} & \hat{V}_{\beta\alpha} & \hat{V}_{\beta\beta} & \hat{V}_{\beta\gamma} & \hat{V}_{\beta\lambda} \\ \hat{V}_{\gamma\nu} & \hat{V}_{\gamma\alpha} & \hat{V}_{\gamma\beta} & \hat{V}_{\gamma\gamma} & \hat{V}_{\gamma\lambda} \\ \hat{V}_{\lambda\nu} & \hat{V}_{\lambda\alpha} & \hat{V}_{\lambda\beta} & \hat{V}_{\lambda\gamma} & \hat{V}_{\lambda\lambda} \end{pmatrix} \right],$$

where $\hat{V}_{ij} = V_{ij}\big|_{\theta=\hat{\theta}}$, $\theta = \theta_i = (\nu, \alpha, \beta, \gamma, \lambda)$ with $[V_{ij}] = [-\ell_{i,j}]^{-1} = [-\partial^2 \ell / (\partial\theta_i \partial\theta_j)]^{-1}$ being the approximate variance covariance matrix. By solving this inverse of dispersion matrix, these solutions will yield the asymptotic variance and covariances of these MLs for $\hat{\nu}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{\lambda}$. Approximate $100(1-a)$ % confidence intervals for $\nu, \alpha, \beta, \gamma$ and λ can be determined respectively as

$$\hat{\nu} \pm z_{\frac{a}{2}} \sqrt{\hat{V}_{\nu\nu}}\,,\ \hat{\alpha} \pm z_{\frac{a}{2}} \sqrt{\hat{V}_{\alpha\alpha}}\,,\ \hat{\beta} \pm z_{\frac{a}{2}} \sqrt{\hat{V}_{\beta\beta}}\,,\ \hat{\gamma} \pm z_{\frac{a}{2}} \sqrt{\hat{V}_{\gamma\gamma}} \text{ and } \hat{\lambda} \pm z_{\frac{a}{2}} \sqrt{\hat{V}_{\lambda\lambda}}$$

where $z_{a/2}$ is the upper a^{th} percentile of the standard normal distribution.

6 Data Analysis

In this section, we use a real data set to show that the ETGIW distribution can be a better model than one based on the TGIW and GIW distributions. The data set given in Table 2 taken from Murthy et al. [7] page 180 and represents 50 items put into use at t = 0 and failure times are in weeks.

Table 2. 50 items put into use at t = 0 and their failure times in weeks

0.013	0.997	4.520	6.572	13.006
0.065	1.284	4.789	7.023	13.388
0.111	1.304	4.849	7.087	13.842
0.111	1.647	5.202	7.291	17.152
0.163	1.829	5.291	7.787	17.283
0.309	2.336	5.349	8.596	19.418
0.426	2.838	5.911	9.388	23.471
0.535	3.269	6.018	10.261	24.777
0.684	3.977	6.427	10.713	32.795
0.747	3.981	6.456	11.658	48.105

The generalized inverse Weibull (GIW), transmuted generalized inverse Weibull (TGIW) and exponentiated transmuted generalized inverse Weibull (ETGIW) distributions are fitted to the data and MLEs of the parameters are given in Table 3. The values of the log-likelihood, Kolmogorove-Smirnov statistic (K-S), Akaike Information Criteria (AIC), and Consistent Akaike Information Criteria (CAIC) for the different fitted distributions are also given, and show that the ETGIW distribution gives a better fit than the others.

Table 3. Maximum likelihood estimates, log-likelihood, K-S, AIC, and CAIC values for the different fitted distributions

Model	ν	α	β	γ	λ	$-\ell$	**K-S**	**AIC**	**CAIC**
GIW	1	0.854	0.479	1.044	0	168.638	0.199	343.276	343.797
TGIW	1	2.383	0.530	1.143	-0.747	166.387	0.192	340.774	341.662
ETGIW	0.502	0.504	0.549	0.962	-0.882	159.999	0.185	329.999	331.363

From Table 3, we observe that the ETGIW distribution is a competitive distribution compared with other distributions. In fact, based on the values of the AIC and CAIC criteria as well as the value of the K-S statistic, we observe that the ETGIW distribution provides the best fit for these data among all the models considered.

7 Conclusion

In this paper, we introduce a new generalization of the inverse Weibull called the exponentiated transmuted generalized inverse Weibull distribution. Some mathematical properties along with estimation issues are addressed. It also provides further flexibility in modeling real data. An application of exponentiated transmuted generalized inverse Weibull distribution to real data show that the new distribution can be used quite effectively to provide better fits than other distributions.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Merovci F, Elbatal I, Ahmed A. The transmuted generalized inverse Weibull distribution. Austrian Journal of Statistics. 2014;43(2):119-131.

[2] Aryal GR, Tsokos CP. On the transmuted extreme value distribution with application. Nonlinear Analysis: Theory, Methods and Applications. 2009;71(12):401-407.

[3] Aryal GR, Tsokos CP. Transmuted Weibull distribution: A generalization of the Weibull probability distribution. European Journal of Pure and Applied Mathematics. 2011;4(2):89-102.

[4] Aryal GR. Transmuted log-logistic distribution. Journal of Statistics Applications & Probability. 2013;2(1):11-20.

[5] Khan MS, King R. Transmuted modified Weibull distribution: A generalization of the modified Weibull probability distribution. European Journal of Pure and Applied Mathematics. 2013;6(1):66-88.

[6] Elbatal I, Muhammed HZ. Exponentiated generalized inverse Weibull distribution. Applied Mathematical Sciences. 2014;8(81):3997-4012.

[7] Murthy DP, Xie M, Jiang R. Weibull models. John Wiley and Sons. 2004;505.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sciencedomain.org/review-history/22806