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A New Criterion for Borel-Euler Summability Method of Fourier Series

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

This paper introduces a new theorem on $(B)(E,1)$ product summability of Fourier series which is the generalization of the result given by Izumi S. [1] under analogous conditions.

Keywords: $(E,1)$ *summability;* $(C,1)(E,1)$ *summability; Borel summability;* $(B)(E,1)$ *summability.*

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1 Introduction

Let $f(x)$ be a function integrable in the sense of Lebesgue over the interval $(-\pi,\pi)$ and periodic with the period 2π , Titchmarsh [2]. Let the Fourier series associated with $f(x)$ be

$$\sum_{n=0}^{\infty} A_n(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1.1}$$

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An infinite series $\sum u_n$ with partial sums S_n are said to be summable $(B)(E,1)$ to s, if

$$\frac{1}{n}\sum_{m=1}^{n} E_m \to s \qquad \text{as } n \to \infty \tag{1.2}$$

Where E_m stands for the $(E,1)$ mean of S_n Hardy, [3].

We shall use the fixed real numbers x and s, the following notations,

$$\phi(t) = f(x+t) + f(x-t) - 2s$$

$$\Phi_1(t) = \int_0^t \phi(u)du$$

$$h(n,t) = \frac{1}{t}\frac{\sin\left\{\frac{p\sin t + t}{2}\right\}}{\exp\left\{p\left(\frac{1-\cos t}{2}\right)\right\}}$$

$$h_1(n,t) = \frac{d}{dt}\frac{1}{t}\left[\frac{\sin\left\{\frac{p\sin t + t}{2}\right\}}{\exp\left\{p\left(\frac{1-\cos t}{2}\right)\right\}}\right]$$

In 1903, Lebesgue H. [4] gave the convergence criteria for Fourier series, at a point x by proving the result.

THEOREM A: If

$$\int_0^t |\phi(u)|du = o(t) \text{ as } t \to 0$$

and

$$\int_{\pi/n}^{\eta} \frac{\left|\phi(t) - \phi\left(t + \frac{\pi}{n}\right)\right|}{t} dt = o(1) \quad \text{as } n \to \infty$$

then the Fourier series of $f(x)$ converges to $f(x)$ at the point x.

Generalising Theorem A for $(C,1)$ summability of Fourier series, Izumi S. [1] proved the following result.

THEOREM B: If

$$\int_0^t \phi(u)du = o(t) \text{ as } t \to 0$$

and

$$\int_{\pi/n}^{\eta} \frac{\left|\phi\left(t+\frac{\pi}{n}\right)-\phi(t)\right|}{t^2} dt = o(1) \quad \text{as } n \to \infty$$

then the Fourier series is summable $(C,1)$ to the sum s .

Recently, Saxena K. [5-6] has proved a theorem for the product summability $(C,1)(E,1)$ of Fourier series under analogous conditions.

THEOREM C: If

$$\int_0^t \phi(u)du = o(t) \text{ as } t \to 0$$

and

$$\int_{\pi/n}^{\eta} \frac{\left|\phi(t)-\phi\left(t+\frac{\pi}{n}\right)\right|}{t^2} \cos^n \frac{t}{2} dt = o(n)$$

as $n \to \infty$, for fixed positive number η >0, then the Fourier series (1.1) is $(C,1)(E,1)$ summable to s at the point x .

Various researchers [7-12] proved some interesting results on summability of Fourier series.

Since, under the conditions of Theorem B, Fourier series is not $(E,1)$ summable to any fixed number, so it is natural to expect the extension of Theorem B for the product summability $(B)(E,1)$ of the Fourier series under analogous conditions.

2 Main Results

In this section, we prove the result by generalizing the conditions of Theorem B for Borel-Euler product summability of Fourier series.

THEOREM 2.1: If

$$\int_0^t \phi(u)du = o(t) \text{ as } t \to 0 \tag{2.1}$$

and

$$\int_{\pi/p}^{\eta} \frac{\left|\phi(t)-\phi\left(t+\frac{\pi}{p}\right)\right|}{t} \exp\left\{-p\left(\frac{1-\cos t}{2}\right)\right\} dt = o(1) \tag{2.2}$$

As $p \to \infty$, for some fixed positive number η >0, then the Fourier series (1.1) is $(B)(E,1)$ summable to s at the point x .

3 Relations

$$\sum_{k=0}^{m}\binom{m}{k}\sin\left(k+\frac{1}{2}\right)t=2^{m}\cos^{m}\frac{t}{2}\sin\left(\frac{m+1}{2}\right)t \tag{3.1}$$

Proof:

$$\sum_{k=0}^{m}\binom{m}{k}\sin\left(k+\frac{1}{2}\right)t=\operatorname{Im}\sum_{k=0}^{m}\binom{m}{k}\exp\left\{i\left(k+\frac{1}{2}\right)t\right\}$$
$$=\operatorname{Im}\exp\left(\frac{it}{2}\right)\{1+\exp(it)\}^{m}$$
$$=2^{m}\cos^{m}\frac{t}{2}\sin\left(m+\frac{1}{2}\right)t$$

Also,

$$\exp(-p)\sum_{m=0}^{\infty}\frac{p^{m}}{m!}\cos^{m}\frac{t}{2}\sin\left(\frac{m+1}{2}\right)t=\exp\left\{-p\left(\frac{1-\cos t}{2}\right)\sin\left(\frac{p\sin t+t}{2}\right)\right\} \tag{3.2}$$

Proof:

$$\exp(-p)\sum_{m=0}^{\infty}\frac{p^{m}}{m!}\cos^{m}\frac{t}{2}\sin\left(\frac{m+1}{2}\right)t=\exp(-p)\operatorname{Im}\sum_{m=0}^{\infty}\frac{p^{m}}{m!}\cos^{m}\frac{t}{2}\exp\left\{i\left(\frac{m+1}{2}\right)t\right\}$$
$$=\exp(-p)\operatorname{Im}\exp\left(\frac{it}{2}\right)\sum_{m=0}^{\infty}\frac{p^{m}}{m!}\cos^{m}\frac{t}{2}\exp\left(\frac{imt}{2}\right)$$
$$=\exp(-p)\operatorname{Im}\exp\left(\frac{it}{2}\right)\exp\left\{p\cos\frac{t}{2}\exp\left(\frac{it}{2}\right)\right\}$$
$$=\exp(-p)\exp\left(p\cos^{2}\frac{t}{2}\right)\sin\left(\frac{p\sin t+t}{2}\right)$$
$$=\exp\left\{-p\left(\frac{1-\cos t}{2}\right)\right\}\sin\left(\frac{p\sin t+t}{2}\right)$$

4 The Estimates

We shall require the following estimates, the first may be verified easily

For $0<t<\dfrac{\pi}{p}$

$$h(n,t)=O(p+1) \tag{4.1}$$

$$\frac{d}{dt}h(n,t)=O\left(\frac{p+1}{2}\right)\frac{1}{t} \tag{4.2}$$

For $t>\frac{\pi}{p}$

$$\exp\left\{-p\sin^2\frac{t}{2}\right\}-\exp\left\{-p\sin^2\left(\frac{t+\frac{\pi}{p}}{2}\right)\right\}=O(t) \tag{4.3}$$

$$\frac{d}{dt}\left[\exp\left(-p\sin^2\frac{t}{2}\right)-\exp\left\{-p\sin^2\left(\frac{t+\frac{\pi}{p}}{2}\right)\right\}\right]=0 \tag{4.4}$$

$$\left(\frac{1}{t}-\frac{1}{t+\frac{\pi}{p}}\right)=o\left(\frac{1}{pt^2}\right) \tag{4.5}$$

$$\frac{d}{dt}\left(\frac{1}{t}-\frac{1}{t+\frac{\pi}{p}}\right)=o\left(\frac{1}{pt^3}\right) \tag{4.6}$$

$$\frac{d}{dt}\left(\frac{1}{t}-\frac{1}{t+\frac{\pi}{p}}\right)\exp\left\{-p\left(\frac{1-\cos t}{2}\right)\right\}\sin\frac{pt}{2}=o\left(\frac{1}{t^2}\right) \tag{4.7}$$

5 Required Lemmas

Lemma 5.1: $\int_{\pi/p}^{(\pi/p)^{\alpha}}\frac{\phi(t)}{t}\frac{\sin(p\sin t+t)-\sin pt}{\exp\left\{\frac{p(1-\cos t)}{2}\right\}}dt=o(1)$

Proof:

Using second mean value theorem and integrating by parts, we have

$$= \int_{\pi/p}^{(\pi/p)^{\alpha}} \frac{\phi(t)}{t} \frac{\sin(p\sin t + t) - \sin pt}{\exp\left\{p\sin^2 \frac{t}{2}\right\}} dt$$

$$= \frac{1}{\exp\left(p\sin^2 \frac{t}{2}\right)} \int_{\pi/p}^{(\pi/p)^{\beta}} \frac{\phi(t)}{t} \{\sin(p\sin t + t) - \sin pt\} dt$$

Since $\frac{1}{3} \leq \alpha \leq \beta < 1$

$$= O(1) \int_{\pi/p}^{(\pi/p)^{\beta}} \frac{\phi(t)}{t} o(pt^3) dt$$

$$= O(p) \int_{\pi/p}^{(\pi/p)^{\beta}} \phi(t)(t^2) dt$$

$$= O(p)\left[o(t)t^2 - 2\int o(t)t dt\right]_{\pi/p}^{(\pi/p)^{\beta}}$$

$$= O(p)\left[o(t^3)\right]_{\pi/p}^{(\pi/p)^{\beta}}$$

$$= o(1), \text{ as } p \to \infty$$

Lemma 5.2:

$$\int_{\pi/p}^{(\pi/p)^{\alpha}} \frac{\phi(t + \pi/p)}{t} \left[\exp\{-p(1-\cos t)\} - \exp\{-p(1-\cos(t+\pi/p))\}\right] \sin pt dt = o(1)$$

Proof:

By mean value theorem of differential calculus,

$$\frac{\exp\{-p(1-\cos(t+\pi/p))\} - \exp\{-p(1-\cos t)\}}{t + \frac{\pi}{p} - t}$$

$$= \frac{d}{dt}\exp\{-p(1-\cos\theta)\} \text{ where } \theta = t + \frac{l\pi}{p};\ (0 < l < 1)$$

$$= -p\sin\theta\exp\{-p(1-\cos\theta)\}$$

$$= -p\sin(t + l\pi/p)\exp\{-p(1-\cos(t + l\pi/p))\}$$

Consequently,

$$\exp\{-p(1-\cos t)\} - \exp\{-p(1-\cos(t+\pi/p))\}$$

$$= \pi\sin(t + l\pi/p)\exp\{-p(1-\cos(t + l\pi/p))\}$$

$$= o(1)$$

Therefore,

$$\left|\int_{\pi/p}^{(\pi/p)^{\alpha}} \frac{\phi(t+\pi/p)}{t}\sin(t+l\pi/p)\exp\{-p(1-\cos(t+l\pi/p))\}\sin pt\right|dt$$

$$=o(1)\int_{\pi/p}^{(\pi/p)^{\alpha}} |\phi(t+\pi/p)|dt$$

$$=o(1),\ \text{as}\ p\to\infty \quad \text{By (2.1)}$$

Lemma 5.3: $\int_{\pi/p}^{(\pi/p)^{\alpha}} \phi(t+\pi/p)\left[\frac{1}{t}-\frac{1}{t+\pi/p}\right]\exp\{-p(1-\cos t)\}\sin pt dt = o(1)$

Proof:

By (2.1) and integrating by parts

$$\leq \frac{\pi}{p}\int_{\pi/p}^{(\pi/p)^{\alpha}} \frac{|\phi(t+\pi/p)|}{t(t+\pi/p)}dt$$

$$< \frac{\pi}{p}\int_{\pi/p}^{(\pi/p)^{\alpha}} \frac{|\phi(t+\pi/p)|}{t^2}dt$$

$$=o\left(\frac{1}{p}\right)\left[o\left(\frac{1}{t^2}\right)o(t+\pi/p)+2\int\frac{o(t+\pi/p)}{t^3}dt\right]_{\pi/p}^{(\pi/p)^{\alpha}}$$

$$=o\left(\frac{1}{p}\right)\left[o\left(\frac{1}{t}\right)+2\int o\left(\frac{1}{t^2}\right)dt\right]_{\pi/p}^{(\pi/p)^{\alpha}}$$

$$=o(1),\ \text{as}\ p\to\infty$$

Lemma 5.4: $\int_{0}^{(\pi/p)} \frac{\phi(t+\pi/p)}{t+\pi/p}\exp\{-p(1-\cos(t+\pi/p))\}\sin pt dt = o(1)$

Proof:

By (2.1) and change of variables,

$$\leq \int_{\pi/p}^{2\pi/p} \frac{|\phi(t)|}{t}dt$$

$$=o\left(\frac{m}{\pi}\right)\int_{\pi/p}^{2\pi/p} |\phi(t)|dt$$

$$=o(1),\ \text{as}\ p\to\infty$$

By using the above-proved lemmas, we will prove the main theorem.

6 Proof of the Main Theorem

Let S_n be the n^{th} partial sum of the Fourier series (1.1), following Zygmund [13], we have

$$S_n - s = \frac{1}{2\pi}\int_0^{\pi}\phi(t)\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

$(E,1)$ transform of S_n is denoted by E_n, we have in relation (3.1)

$$E_n = \frac{1}{2\pi}2^{-n}\int_0^{\pi}\frac{\phi(t)}{\sin\frac{t}{2}}\sum_{k=0}^{n}\binom{n}{k}\sin\left(k+\tfrac{1}{2}\right)t\,dt$$

$$= \frac{1}{2\pi}\int_0^{\pi}\frac{\phi(t)}{\sin\frac{t}{2}}\cos^m\tfrac{t}{2}\sin\left(\tfrac{m+1}{2}\right)t\,dt \qquad (6.1)$$

Superimposing Borel transform on E_n, hence by relation (3.2), we find

$$B_p = \frac{1}{2\pi}e^{-p}\int_0^{\pi}\frac{\phi(t)}{\sin\frac{t}{2}}\sum_{m=0}^{\infty}\frac{p^m}{m!}\cos^m\tfrac{t}{2}\sin\left(\tfrac{m+1}{2}\right)t\,dt$$

$$= \frac{1}{2\pi}\int_0^{\pi}\frac{\phi(t)}{\sin\frac{t}{2}}\exp\left\{-p\left(\frac{1-\cos t}{2}\right)\right\}\sin\left(\frac{p\sin t + t}{2}\right)dt$$

which may also be written as

$$B_p = \frac{1}{\pi}\int_0^{\pi}\frac{\phi(t)}{t}\exp\left\{-p\left(\frac{1-\cos t}{2}\right)\right\}\sin\left(\frac{p\sin t + t}{2}\right)dt$$

$$= \frac{1}{\pi}\int_0^{\delta}\frac{\phi(t)}{t}\exp\left\{-p\left(\frac{1-\cos t}{2}\right)\right\}\sin\left(\frac{p\sin t + t}{2}\right)dt + o(1)$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/p} + \int_{\pi/p}^{(\pi/p)^{\alpha}} + \int_{(\pi/p)^{\alpha}}^{\delta}\right]\frac{\phi(t)}{t}\exp\left\{-p\left(\frac{1-\cos t}{2}\right)\right\}\sin\left(\frac{p\sin t + t}{2}\right)dt$$

$$= \left[I_1 + I_2 + I_3\right] + o(1), \quad \text{(say)}$$

Integrating by parts and using estimates (4.1), (4.2) given conditions

$$I_1 = \frac{1}{\pi}\int_0^{\pi/p} \frac{\phi(t)}{t} \exp\left\{-p\left(\frac{1-\cos t}{2}\right)\right\}\sin\left(\frac{p\sin t + t}{2}\right)dt$$

$$= \frac{1}{\pi}\int_0^{\pi/p} \phi(t)h(n,t)dt$$

$$= \frac{1}{\pi}\left[\Phi_1(t)h(n,t) - \int \frac{d}{dt}h(n,t)\Phi_1(t)dt\right]_0^{\pi/p}$$

$$= \frac{1}{\pi}\left[\{o(t)O(p)\}_0^{\pi/p} - \int_0^{\pi/p} o(t)O\left(\frac{p}{t}\right)dt\right]$$

$$= o(1) + o(p)\int_0^{\pi/p} dt$$

$$= o(1) \quad \text{as } p \to \infty$$

Using Lemma

$$2I_2 = \frac{1}{\pi}\int_{\pi/p}^{(\pi/p)^\alpha} \frac{\phi(t)}{t}\exp\{-p(1-\cos t)\}\sin ptdt$$

$$-\frac{1}{\pi}\int_0^{(\pi/p)^\alpha - \pi/p} \frac{\phi(t+\pi/p)}{(t+\pi/p)}\exp\{-p(1-\cos t)\}\sin ptdt$$

$$= \frac{1}{\pi}\int_{\pi/p}^{(\pi/p)^\alpha} \frac{\phi(t)-\phi(t+\pi/p)}{t}\exp\{-p(1-\cos t)\}\sin ptdt$$

$$+\frac{1}{\pi}\int_{\pi/p}^{(\pi/p)^\alpha} \frac{\phi(t+\pi/p)}{t}\left[\exp\{-p(1-\cos(t+\pi/p))\} - \exp\{-p(1-\cos t)\}\right]\sin ptdt$$

$$+\frac{1}{\pi}\int_{\pi/p}^{(\pi/p)^\alpha} \phi(t+\pi/p)\left[\frac{1}{t} - \frac{1}{t+\pi/p}\right]\exp\{-p(1-\cos t)\}\sin ptdt$$

$$-\frac{1}{\pi}\int_0^{(\pi/p)} \frac{\phi(t+\pi/p)}{t+\pi/p}\exp\{-p(1-\cos(t+\pi/p))\}\sin ptdt$$

$$+\frac{1}{\pi}\int_{(\pi/p)^\alpha - \pi/p}^{(\pi/p)^\alpha} \frac{\phi(t+\pi/p)}{t+\pi/p}\exp\{-p(1-\cos(t+\pi/p))\}\sin ptdt$$

$$= e_1 + e_2 + e_3 + e_4 + e_5$$

By given condition, we get

$$|e_1| \le \frac{1}{\pi}\int_{\pi/m}^{(\pi/m)^\alpha} \frac{|\phi(t)-\phi(t+\pi/p)|}{t}\exp\{-p(1-\cos t)\}dt$$

$$= o(1) \quad \text{as } p \to \infty$$

Applying Lemma (5.2), (5.3) and (5.4), we get

$$e_2 = e_3 = e_4 = o(1)$$

Lastly considering e_5, we have by change of variables

$$e_5 = \frac{1}{\pi}\int_{(\pi/p)^\alpha}^{(\pi/p)^\alpha+\pi/p} \frac{\phi(t)}{t}\exp\{-p(1-\cos t)\}\sin ptdt$$

$$|e_5| \le \frac{1}{\pi}\int_{(\pi/p)^\alpha}^{(\pi/p)^\alpha+\pi/p} \frac{|\phi(t)|}{t}dt$$

$$\le \frac{1}{\pi}\left(\frac{p}{\pi}\right)^\alpha \int_{(\pi/p)^\alpha}^{(\pi/p)^\alpha+\pi/p} |\phi(t)|dt$$

$$= O\left(\frac{1}{p^{-\alpha}}\right)$$

$$= o(1) \quad \text{as } p \to \infty$$

By the continuity parts of $\int|\phi(t)|dt$ and the function $\alpha < 1/2$

Thus, $I_2 = o(1)$

Now, at last, we have

$$I_3 = \frac{1}{\pi}\int_{(\pi/p)^\alpha}^{\delta} \frac{\phi(t)}{t}\exp\left\{-p\left(\frac{1-\cos t}{2}\right)\right\}\sin\left(\frac{p\sin t + t}{2}\right)dt$$

$$= \frac{1}{\pi}\int_{(\pi/p)^\alpha}^{\delta} \frac{\phi(t)}{t}\frac{\sin(p\sin t + t)}{\exp\{p\sin^2 \frac{t}{2}\}}dt$$

$$\le \frac{1}{\pi}\frac{(p/\pi)^\alpha}{\exp\{p\sin^2 \left(\frac{\pi}{p}\right)^\alpha\}}\int_{(\pi/p)^\alpha}^{\delta} \phi(t)\sin(p\sin t + t)dt$$

$$\le \frac{p^\alpha}{\exp\{p^{1-2\alpha}\}}\int_{(\pi/p)^\alpha}^{\delta} |\phi(t)|dt$$

$$= o(1) \quad \text{as } p \to \infty$$

This completes the proof of theorem 2.1.

7 Conclusion

In this paper, we have introduced the product summability of Fourier series using Borel-Euler summation method. The present theorem extends, generalizes and improves many existing results on summability of Fourier series and its allied series. This result may be a motivation to other researchers to carry out the outcomes in the field of summability theory.

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Competing Interests

Authors have declared that no competing interests exist.

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