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Vectorial Iterative Fractional Laplace Transform Method for the Analytic Solutions of Fractional Cauchy-Riemann Systems Partial Differential Equations

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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Abstract

The present study aims to obtain infinite fractional power series solution vectors of fractional Cauchy-Riemann systems equations with initial conditions by the use of vectorial iterative fractional Laplace transform method (VIFLTM). The basic idea of the VIFLTM was developed successfully and applied to four test examples to see its effectiveness and applicability. The infinite fractional power series form solutions were successfully obtained analytically. Thus, the results show that the VIFLTM works successfully in solving fractional Cauchy-Riemann system equations with initial conditions, and hence it can be extended to other fractional differential equations.

Keywords: Fractional Cauchy-Riemann systems equations; Caputo fractional derivatives; vectorial iterative fractional Laplace transform method.

1 Introduction

Fractional calculus theory is a mathematical analysis tool to the study of arbitrary order integrals and derivatives, which unify and generalize the notations of integer-order differentiation and $n-$ fold integration

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[1,2,3,4]. The L'Hopital's letter raised the question "What does $\frac{\partial^m f(x)}{\partial x^m}$ mean if $m=\frac{1}{2}$?" to Leibniz in 1695 is considered to be where the idea of fractional calculus started [2,5,6,7,8]. Since then, so many works on this question and other related questions have done up to the middle of the 19th century by many famous mathematicians such as Laplace, Fourier, Abel, Liouville, Riemann, Grunwald, Letnkov, Levy, Marchaud, Erdelyi and Reisz and these works sum up leads to contributions creating the field which is known today as fractional calculus [3].

However, fractional calculus has almost the age of standard calculus, it was only in recent few decades that its theory and applications have rapidly developed. Ross [9] was the first in organizing the first international conference on fractional calculus and its applications at the University of New Haven in June 1974 and edited the proceedings, and Oldham and Spanier [3] published the first monograph on fractional calculus in 1974. Then after, because of the fact that fractional derivatives and integrals are non-local operators and then this property makes them a powerful instrument for the description of memory and hereditary properties of different substances [4], theory and applications of fractional calculus have attracted much interest and become an exciting research area. For more detailed information, the reader is kindly requested to go through [10,11] to know more details about the mathematical properties of fractional derivatives and fractional integrals, including their types and history, their motivation for use, their characteristics, and their applications.

Due to this, fractional calculus has got many important applications in different fields of science, engineering and finance. For example, Shanantu Das [12] discussed that fractional calculus is applicable to problems in: fractance circuits, electrochemistry, capacitor theory, feedback control system, vibration damping system, diffusion process, electrical science, and material creep. Podlubny [4] discussed that fractional calculus is applicable to problems in fitting experimental data, electric circuits, electro-analytical chemistry, fractional multi-poles, neurons and biology [4]. Fractional calculus is also applicable to problems in: polymer science, polymer physics, biophysics, rheology, and thermodynamics [6]. In addition, it is applicable to problems in: electrochemical process [2,3,4], control theory [4,13], physics [14], science and engineering [8], transport in semi-infinite medium [3], signal processing [15], food science [16], food gums [17], fractional dynamics [18,19], modeling Cardiac tissue-electrode interface [20], food engineering and econophysics [13], complex dynamics in biological tissues [21], viscoelasticity [4,14,16,22,10], modeling oscillation systems [23]. Some of these mentioned applications were tried to be touched as follows.

In the area of science and engineering, different applications of fractional calculus have been developed in the last two decades. For instance, fractional calculus was used in image processing, mortgage, biosciences, robotics, the motion of fractional oscillator and analytical science [8]. It was also used to generalize traditional classical inventory model to fractional inventory model [24].

In the area of the electrochemical process, for example, half-order derivatives and integrals proved to be more useful for the formulation of certain electrochemical problems than the classical models [2,3,4].

In the area of viscoelasticity, the use of fractional calculus for modeling viscoelastic materials is well known. For viscoelastic materials, the stress-strain constitutive relation can be more accurately described by introducing the fractional derivative [4,22,10,25,26,27]. Fractional derivatives, which are the one part of fractional calculus, are used to name derivatives of an arbitrary order [4]. Recently, fractional derivatives have been successfully applied to describe (model) real-world problems.

In the area of physics, fractional kinetic equations of the diffusion, diffusion-advection and Fokker-Plank type are presented as a useful approach for the description of transport dynamics in complex systems that are governed by anomalous diffusion and non-exponential relaxation patterns [28]. Metzler and Klafter [28] derived these fractional equations asymptotically from basic random walk models, and from a generalized master equation. They presented an integral transformation between the Brownian solution and its fractional counterparts. Moreover, a phase space model was presented to explain the genesis of fractional dynamics in trapping systems. These issues make the fractional equation approach powerful. Their work demonstrates

that the fractional equations have come of age as a complementary tool in the description of anomalous transport processes. L.R. Da Silva, Tateishi, M.K. Lenzi, Lenzi and Da silva [29] were also discussed that solutions for a system governed by a non-Markovian Fokker Planck equation and subjected to a Comb structure were investigated by using the Green function approach. This structure consists of the axis of the structure as the backbone and fingers which are attached perpendicular to the axis, and for this system, an arbitrary initial condition in the presence of time-dependent diffusion coefficients and spatial fractional derivatives were considered and the connection to the anomalous diffusion was analyzed [29].

In addition to these, the following are also other applications of fractional derivatives. Fractional derivatives in the sense of Caputo fractional derivatives were used in generalizing some theorems of classical power series to fractional power series [1]. Caputo time-fractional derivatives was used to model a natural convection flow; see [30]. Fractional derivative was applied to study the effects of Lorentz force induced by convection; see [31] Fractional derivative in the Caputo sense was used to introduce a general form of the generalized Taylor's formula by generalizing some theorems related to the classical power series into fractional power series sense [32]. A definition of Caputo fractional derivative proposed on a finite interval in the fractional Sobolev spaces was investigated from the operator theoretic viewpoint [33]. Particularly, some important equivalence of the norms related to the fractional integration and differentiation operators in the fractional Sobolev spaces are given and then applied for proving the maximal regularity of the solutions to some initial-boundary-value problems for the time-fractional diffusion equation with the Caputo derivative in the fractional Sobolev spaces [33]. With the help of Caputo time-fractional derivative and Riesz space-fractional derivative, the β-fractional diffusion equation, which is a special model for the two-dimensional anomalous diffusion, is deduced from the basic continuous time random walk equations in terms of a time- and space-fractional partial differential equation with the Caputo time-fractional derivative of order $\frac{\beta}{2}$ and the Riesz space-fractional derivative of order β [34]. Fractional derivatives were also used to describe HIV infection of $CD4^{+}T$ with therapy effect [35].

In the area of modelling oscillating systems, caputo and Caputo-Fabrizio fractional derivatives were used to present fractional differential equations which are generalization of the classical mass-spring-damper model, and these fractional differential equations are used to describe a variety of systems which had not been addressed by the classical mass-spring-damper model due to the limitations of the classical calculus [23].

Podlubny [4] stated that fractional differential equations are equations which contain fractional derivatives. These equations can be divided into two categories such as fractional ordinary differential equations and fractional partial differential equations. Fractional partial differential equations (PDES) are a type of differential equations (DEs) that involving multivariable function and their fractional or fractional-integer partial derivatives with respect to those variables [36]. There are different examples of fractional partial differential equations. Some of them are: the time-fractional Boussinesq-type equation, the time-fractional $B(2,1,1)$-type equation and the time-fractional Klein-Gordon-type equation stated in Abu Arqub et al. [36], and time fractional diffusion equation stated in A. Kumar, Kumar and Yan [37], Cetinkaya and Kiymaz [38], Kumar, Yildirim, Khan and Wei [39], Kebede [40,41] and so on.

Recently, fractional differential equations have been successfully applied to describe (model) real-world problems. For instance, the generalized wave equation, which contains fractional derivatives with respect to time in addition to the second-order temporal and spatial derivatives, was used to model the viscoelastic case and the pure elastic case in a single equation [42]. The time fractional Boussinesq-type equations can be used to describe small oscillations of nonlinear beams, long waves over an even slope, shallow-water waves, shallow fluid layers, and nonlinear atomic chains; the time-fractional $B(2,1,1)$-type equations can be used to study optical solitons in the two-cycle regime, density waves in traffic flow of two kinds of vehicles, and surface acoustic soliton in a system supporting love waves; the time fractional Klein-Gordon-type equations can be applied to study complex group velocity and energy transport in absorbing media, short waves in nonlinear dispersive models, propagation of dislocations within crystals as cited in [43]. As cited in Abu

Arqub [43], the time-fractional heat equation, which is derived from Fourier's law and conservation of energy, is used in describing the distribution of heat or variation in temperature in a given region over time; the time-fractional cable equation, which is derived from the cable equation for electro diffusion in smooth homogeneous cylinders and occurred due to anomalous diffusion, is used in modelling the ion electro diffusion at the neurons; the time-fractional modified anomalous sub diffusion equation, which is derived from the neural cell adhesion molecules, is used for describing processes that become less anomalous as time progresses by the inclusion of a second fractional time derivative acting on the diffusion term; the time fractional reaction sub diffusion equation is used to describe many different areas of chemical reactions, such as exciton quenching, recombination of charge carriers or radiation defects in solids, and predator pray relationships in ecology; the time-fractional Fokker–Planck equation is used to describe many phenomena in plasma and polymer physics, population dynamics, neurosciences, nonlinear hydrodynamics, pattern formation, and psychology; the time-fractional Fisher's equation is used to describe the population growth models, whilst, the time fractional Newell–Whitehead equation is used to describe fluid dynamics model and capillary–gravity waves [41,44]. The fractional differential equations, a generalization of the classical mass-spring-damper models, are useful to understand the behaviour of dynamical complex systems, mechanical vibrations, control theory, relaxation phenomena, viscoelasticity, viscoelastic damping and oscillatory processes [23]. The space-time fractional diffusion equations on two-time intervals were used in finance to model option pricing and the model was shown to be useful for option pricing during some temporally abnormal periods [45]. The β-fractional diffusion equation for $0 < \beta < 2$ describes the so called Levy flights that correspond to the continuous time random walk model, where both the mean waiting time and the jump length variance of the diffusing Particles are divergent [34]. Time fractional diffusion equations in the Caputo sense with initial conditions are used to model cancer tumor [46].

The system of first-order linear equations:

$$\frac{\partial}{\partial y}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} + \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix};\ x \in IR, y > 0 \tag{1.a}$$

for the desired vector $\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ is involving real-valued functions $u(x,y)$ and $v(x,y)$. Together, System (1.a) is elliptic while individually both the partial differential equations are hyperbolic for the ellipticity of the system [47]. If $f(x,y) = 0 = g(x,y)$, Equation (1.a) is the Cauchy-Riemann system and dependent variables u and v are analytic. Thinking of y as a time variable and of data for the vector $\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ as being given on $y = 0$, we are mainly concerned with the inhomogeneous Cauchy-Riemann System (1.a) subject to the following initial condition [48]:

$$\begin{bmatrix} u_0(x,y) \\ v_0(x,y) \end{bmatrix} = \begin{bmatrix} \varphi(x) \\ \psi(x) \end{bmatrix};\ x \in IR \tag{1.b}$$

where $\phi(x)$ and $\psi(x)$ are analytic.

It is well known that initial value problem for the Cauchy-Riemann system is ill-posed. The inherent instability of this system, for the first time was discussed by Hadamard [49]. Farmer and Howison [50] illustrate the ill-posed nature of the system in various contexts.

Joseph and Saut [51] associated that the ill-posedness of Cauchy problem with the non-existence of a solution to the initial-value problem for non-analytic data. They showed that the problems which are Hadamard unstable cannot be solved unless the initial data are analytic. Reichel [52] analyzed several fast numerical methods based on solving initial-value problems for the Cauchy-Riemann system. She discussed the techniques for analytic continuation of conformal mappings and indicated the available methods for finding analytic continuations which use Taylor coefficients or their approximations for the analytic functions, see for example Gustafson [53] and Henrici [54]. Reichel [52] also shows the stability and accuracy of her schemes through numerous applications.

During the past couple of decades, researchers have been engrossed to constructing the approximate analytic solution for the partial differential equations. For example, Naseem and Tahir [48] used a Vectorial reduced differential transform (VRDT) method to solve the initial-value problem for the inhomogeneous Cauchy-Riemann system.

Recently, beyond standard partial differential equations, the fractional differential equations have gained much attention of researchers due to the fact that they generate fractional Brownian motion which is a generalization of Brownian motion [4]. Due this, fractional derivative provides an excellent tool for describing memory and hereditary properties for various [4] materials and processes ordinary (standard derivative) [55], and then based on this fact, the systems of first order linear equations form given by (1.a) [48] were extended to the form:

$$\begin{cases} D_y^{\beta}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \dfrac{\partial}{\partial x}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} + \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}; \; p-1 < \beta \le p, x, y \in IR, \; y > 0 & (2.a) \\ \begin{bmatrix} u_0^r \\ v_0^r \end{bmatrix} = \begin{bmatrix} f_r(x,0) \\ g_r(x,0) \end{bmatrix}; \; r = 0, 1, 2, \cdots, p-1; x \in IR & (2.\text{b}) \end{cases}$$

where $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$ is the source terms, which is a generalization of equation $(1.a)$ given $(1.b)$, was considered and then solved it by Vectorial Iterative Fractional Laplace Transform method in this paper. Here $D_y^{\beta}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = J_y^{1-\beta}\begin{bmatrix} \dfrac{\partial}{\partial y}u(x,y) \\ \dfrac{\partial}{\partial y}v(x,y) \end{bmatrix}$, where the Caputo fractional derivative, D_y^{β} has the advantage that the initial conditions for fractional differential equations with Caputo derivative take on the same form as for integer order differential equations [56].

The iterative method was firstly introduced by Daftardar-Gejji and Jafari [57] to solve numerically the nonlinear functional equations. By now, the iterative method has been used to solve many integer and fractional boundary value problem [58]. Jafari et al. [59] firstly solved the fractional partial differential equations by the use of iterative Laplace transform method (ILTM). More recently, Yan [60], Sharma and Bairwa [61], Sharma and Bairwa [62], Kebede [41] were used VILTM for solving Fractional Fokker-Planck equations, generalized time-fractional biological population model, Fractional Heat and Wave-Like Equations, and (n+1)-Dimensional time fractional diffusion equations respectively.

In this paper, the author has been presented how to obtain the solutions of fractional Cauchy-Riemann systems with initial conditions in the form infinite fractional power series form by the use of vectorial iterative fractional Laplace transform method (VIFLTM). The basic idea of VIFLTM was developed successfully, and then four test examples were presented to see its effectiveness and applicability. Their

solutions in the form of infinite fractional power series were successfully obtained by the use of vectorial iterative fractional Laplace transform method (VIFLTM). The results show that the vectorial iterative fractional Laplace transform method works successfully in solving fractional Cauchy-Riemann systems, and hence it can be extended to other systems of equations.

This paper is organized as follows in the next sections. In the methods and materials section, the way the study was designed to go through was discussed. In the preliminaries section, some definitions, properties and theorems of fractional calculus theory. In the results and discussion section, the results which are the basic idea of vectorial fractional Laplace transform method, application examples and discussion of the application of the results obtained were presented. Finally, the conclusions are presented.

2 Methods and Materials

First, the background theory for the objective of the study was set. Next, inhomogeneous Cauchy system of fractional differential equations with initial conditions of the form: Equation $(2.a)$ given that Equation $(2.b)$ was considered and then solved analytically by using vectorial iterative fractional Laplace transform method following the next five procedures sequentially. First, some basic definitions and properties of fractional calculus and Laplace transform were revisited. Secondly, basic idea of iterative fractional Laplace transform method for Equation $(8.a)$ given that Equation $(8.b)$ was developed and then remark 3.2.2.1 was obtained. Thirdly, solutions of Equation $(2.a)$ given that Equation $(2.b)$ in the form of infinite fractional power series was obtained by using the remark 3.2.2.1. Lastly, the exact solution vector of the standard form of Equations $(2.a)$ given that Equation $(2.b)$ for the special case $\beta = 1$ was obtained.

3 Preliminaries

Some basic definitions and properties of fractional calculus and Laplace transform were revisited as follows to use them in this paper; see [2,4,10,11].

Definition3.1. A real valued function $u(x,t), x \in IR, t > 0$, is said to be in the space C_μ, $\mu \in IR$, if there exists a real number $q > \mu$ such that $u(x) = t^q u_1(x,t)$, where $u_1(x,t) \in C(IR \times [0,+\infty))$ and it is said to be in the space C_μ^m if $u^{(m)}(x,t) \in C_\mu, n \in IN$.

Definition3.2. The Riemann-Liouville fractional integral operator of order $\beta \geq 0$ of a function $u(x,t) \in C_\mu, \mu > -1$ is defined as

$$J_t^\beta u(x,t) = \begin{cases} \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^t (x-\xi)^{\beta-1} u(x,\xi) d\xi, \ 0 < \xi < \mathrm{t}, \beta > 0 \\ u(x,t), \beta = 0 \end{cases} \tag{3}$$

Consequently, for $\alpha, \beta \geq 0, C \in IR, u(x,t) \in C_\mu^m, u(x,t) \in C_\mu, \mu > -1$, the operator J_t^β has the properties $J_t^\alpha J_t^\beta u(x,t) = J_t^{\alpha+\beta} u(x,t) = J_t^\beta J_t^\alpha u(x,t)$, $J_t^\beta c = \left(\dfrac{c}{\Gamma(\alpha+1)} \right) t^\beta$ and $J_t^\beta t^\gamma = \left(\dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\beta)} \right) t^{\gamma+\beta}$.

But, Riemann Liouville derivative does not allow the utilization of initial and boundary conditions involving integer order derivatives during modelling real-world problems with fractional differential equations. To beat this disadvantage of Riemann Liouville derivative [2,4], Caputo proposed a modified fractional differentiation operator D_a^β [55] to illustrate the theory of viscoelasticity as follows:

$$D_a^\beta f(x) = J_a^{m-\beta} D^m f(x) = \frac{1}{\Gamma(m-\beta)} \int_a^x (x-\xi)^{m-\beta-1} f^{(m)}(\xi) d\xi, \beta \geq 0 \tag{4}$$

where $m-1 < \beta < m, x > a$ and $f \in C_{-1}^m$.

This Caputo fractional derivative allows the utilization of initial and boundary conditions involving integer order derivatives, which have clear physical interpretations of the real situations.

Definition3.3. For the smallest integer that exceeds β , the Caputo time fractional derivative order $\beta > 0$ of a function $u(x,t)$ is defined as**:**

$$D_t^\beta u(x,t) = \begin{cases} \frac{1}{\Gamma(m-\beta)} \int_0^t (t-\xi)^{m-\beta-1} \frac{\partial^m u(x,\xi)}{\partial \xi^m} d\xi \\ \frac{\partial^m u(x,t)}{\partial t^m}, \beta = m \end{cases} = J^{m-\beta} \frac{d^m}{dt^m} u(x,t), \ 0 \leq m-1 < \beta < m \tag{5}$$

Theorem3.1. If $m-1 < \beta \leq m$, $\forall m \in \mathrm{IN}, u(x,t) \in C_\gamma^m, \gamma \geq -1$ then

i. $D_t^\beta J_t^\beta u(x,t) = u(x,t)$.

ii. $J^\beta D^\beta u(x,t) = u(x,t) - \sum_{k=0}^{m-1} \frac{\partial^k}{\partial t^k} u(x,0^+) \frac{t^k}{k!}, \ t > 0$.

Definition3.4. [11] Laplace transform of $\phi(t), t > 0$ is

$$L[f(t)] = F(s) = \int_0^\infty e^{-st} f(t) dt \tag{6}$$

Definition3.5. [11] Laplace transform of $D_t^\beta u(x,t)$ is

$$L[D_t^\beta u(x,t)] = L[u(x,t)] - \sum_{k=1}^{q-1} u^k(x,0) s^{\beta-k-1}, q-1 < \beta \leq q, q \in IN \tag{7}$$

4 Results and Discussion

4.1 Main results

Basic idea of vectorial Iterative fractional Laplace transform method is illustrated as follows.

Step1. Consider a fractional non-linear inhomogeneous Cauchy Riemann System of partial differential equations with initial conditions of the form:

$$\begin{cases} D_y^{\beta}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial x}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} - \begin{bmatrix} Nu(x,y) \\ Nv(x,y) \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}; \; p-1 < \beta \le p & (8.a) \\ \begin{bmatrix} u_0^r(x,0) \\ v_0^r(x,0) \end{bmatrix} = \begin{bmatrix} f_r(x,0) \\ g_r(x,0) \end{bmatrix}; r = 0,\, 1,\, 2, \cdots,\, p-1 & (8.b) \end{cases}$$

where $D_y^{\beta}\begin{bmatrix} u \\ v \end{bmatrix} = D_y^{\beta}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}, \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}, D_y^{\beta}\begin{bmatrix} u \\ v \end{bmatrix}$ is the Caputo fractional derivative of $\begin{bmatrix} u \\ v \end{bmatrix}$, N is general nonlinear operator and $\begin{bmatrix} f \\ g \end{bmatrix}$ is the source terms.

Step2. Now apply fractional Laplace transform method to Equation $(8.a)$ given that Equation $(8.b)$ as follows.

i. Applying the Laplace transform denoted by L to Equation $(8.a)$, we obtain:

$$L\left(D_y^{\beta}\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}\right) - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} L\left(\frac{\partial}{\partial x}u\right) \\ L\left(\frac{\partial}{\partial x}v\right) \end{bmatrix} + L\begin{bmatrix} Nu \\ Nv \end{bmatrix} = \begin{bmatrix} L[f] \\ L[g] \end{bmatrix} \quad (9)$$

ii. By using Equation (7), we get:

$$L\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \frac{1}{s^{\beta}}\left(\sum_{r=0}^{p-1} u^r(x,o)s^{\beta-r-1} + L[f]\right) \\ \frac{1}{s^{\beta}}\left(\sum_{r=0}^{p-1} v^r(x,0)s^{\beta-r-1} + L[g]\right) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{S^{\beta}}L\left(\frac{\partial}{\partial x}u\right) \\ \frac{1}{S^{\beta}}L\left(\frac{\partial}{\partial x}v\right) \end{bmatrix} + \begin{bmatrix} \frac{1}{S^{\beta}}L(Nu) \\ \frac{1}{S^{\beta}}L(\mathrm{Nv}) \end{bmatrix} \quad (10)$$

iii. Taking inverse Laplace transform of Equation (10), we get:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} L^{-1}\left(\frac{1}{s^{\beta}}\left(\sum_{r=0}^{p-1} u^r(x,o)s^{\beta-r-1} + L[f]\right)\right) \\ L^{-1}\left(\frac{1}{s^{\beta}}\left(\sum_{r=0}^{p-1} v^r(x,0)s^{\beta-r-1} + L[g]\right)\right) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} L^{-1}\left(\frac{1}{S^{\beta}}L\left(\frac{\partial}{\partial x}u\right)\right) \\ L^{-1}\left(\frac{1}{S^{\beta}}L\left(\frac{\partial}{\partial x}v\right)\right) \end{bmatrix} + \begin{bmatrix} L^{-1}\left(\frac{1}{S^{\beta}}L(Nu)\right) \\ L^{-1}\left(\frac{1}{S^{\beta}}L(Nv)\right) \end{bmatrix} \quad (11)$$

Step3. Now we apply the iterative method to Equation (11) as follows.

i. Let $\begin{bmatrix} u \\ v \end{bmatrix}$ be the solution vector of Equation $(8.a)$ and has the infinite series form

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{\infty} u_i \\ \sum_{i=0}^{\infty} v_i \end{bmatrix} \text{ ; where } \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}, \quad \begin{bmatrix} \sum_{i=0}^{\infty} u_i \\ \sum_{i=0}^{\infty} v_i \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{\infty} u_i(x,y) \\ \sum_{i=0}^{\infty} v_i(x,y) \end{bmatrix} \tag{12}$$

Since, $\frac{\partial}{\partial x}$ is the linear operator, using Equation (12),

$$\frac{\partial}{\partial x}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} u \\ \frac{\partial}{\partial x} v \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} \sum_{i=0}^{\infty} u_i \\ \frac{\partial}{\partial x} \sum_{i=0}^{\infty} v_i \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{\infty} \frac{\partial}{\partial x} u_i \\ \sum_{i=0}^{\infty} \frac{\partial}{\partial x} v_i \end{bmatrix} \tag{13}$$

ii. Since N is the non-linear operator, by using Equation (12), N is decomposed as:

$$\begin{bmatrix} Nu \\ Nv \end{bmatrix} = \begin{bmatrix} N\left(\sum_{i=1}^{\infty} u_i\right) \\ N\left(\sum_{i=1}^{\infty} v_i\right) \end{bmatrix} = \begin{bmatrix} N(u_0) + \sum_{i=0}^{\infty} \left(N\left(\sum_{r=0}^{i} u_r\right) - N\left(\sum_{r=0}^{i-1} u_r\right)\right) \\ N(v_0) + \sum_{i=0}^{\infty} \left(N\left(\sum_{r=0}^{i} v_r\right) - N\left(\sum_{r=0}^{i-1} v_r\right)\right) \end{bmatrix} \tag{14}$$

iii. By substituting Equations (12), (13) and (14) in Equation (11), we get

$$\begin{bmatrix} \sum_{i=1}^{\infty} u_i \\ \sum_{i=1}^{\infty} v_i \end{bmatrix} = \begin{bmatrix} L^{-1}\left(\frac{1}{S^{\beta}}\left(\sum_{r=0}^{p-1} u^{r} s^{\beta-r-1} + L[f]\right)\right) \\ L^{-1}\left(\frac{1}{S^{\beta}}\left(\sum_{r=0}^{p-1} v^{r} s^{\beta-r-1} + L[g]\right)\right) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L^{-1}\left(\frac{1}{S^{\beta}} L\left(\sum_{i=0}^{\infty} \frac{\partial}{\partial x} u_i\right)\right) \\ L^{-1}\left(\frac{1}{S^{\beta}} L\left(\sum_{i=0}^{\infty} \frac{\partial}{\partial x} v_i\right)\right) \end{bmatrix} + \begin{bmatrix} L^{-1}\left(\frac{1}{S^{\beta}} L\left(N(u_0) + \sum_{i=1}^{\infty}\left(N\left(\sum_{r=0}^{i} u_j\right) - N\left(\sum_{r=0}^{i-1} u_r\right)\right)\right)\right) \\ L^{-1}\left(\frac{1}{S^{\beta}} L\left(N(v_0) + \sum_{i=1}^{\infty}\left(N\left(\sum_{r=0}^{i} v_j\right) - N\left(\sum_{r=0}^{i-1} v_r\right)\right)\right)\right) \end{bmatrix} \tag{15}$$

iv. Now from Equation (15), we define recurrence relations:

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = L^{-1}\begin{bmatrix} L^{-1}\left(\frac{1}{S^{\beta}}\left(\sum_{r=0}^{p-1} u^{r} s^{\beta-r-1} + L[f]\right)\right) \\ L^{-1}\left(\frac{1}{S^{\beta}}\left(\sum_{r=0}^{p-1} v^{r} s^{\beta-r-1} + L[f]\right)\right) \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L^{-1}\left(\frac{1}{S^{\beta}} L\left(\frac{\partial}{\partial x}(u_0)\right)\right) \\ L^{-1}\left(\frac{1}{S^{\beta}} L\left(\frac{\partial}{\partial x}(v_0)\right)\right) \end{bmatrix} + \begin{bmatrix} L^{-1}\left(\frac{1}{S^{\beta}} L\left(N(u_0)\right)\right) \\ L^{-1}\left(\frac{1}{S^{\beta}} L\left(N(v_0)\right)\right) \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_{1+1} \\ v_{1+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L^{-1}\left(\frac{1}{S^{\beta}} L\left(\frac{\partial}{\partial x}\left(\sum_{i=0}^{1} u_i - u_0\right)\right)\right) \\ -1\left(\frac{1}{S^{\beta}} L\left(\frac{\partial}{\partial x}\left(\sum_{i=0}^{1} v_i - v_0\right)\right)\right) \end{bmatrix} + \begin{bmatrix} L^{-1}\left(\frac{1}{S^{\beta}} L\left(N\left(\sum_{i=0}^{1} u_i\right) - N(u_0)\right)\right) \\ -1\left(\frac{1}{S^{\beta}} L\left(N\left(\sum_{i=0}^{1} v_i\right) - N(v_0)\right)\right) \end{bmatrix} \tag{18}$$

$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_{2+1} \\ v_{2+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L^{-1}\left(\frac{1}{S^{\beta}} L\left(\frac{\partial}{\partial x}\left(\sum_{i=0}^{2} u_i - \sum_{i=0}^{1} u_i\right)\right)\right) \\ L^{-1}\left(\frac{1}{S^{\beta}} L\left(\frac{\partial}{\partial x}\left(\sum_{i=0}^{2} u_i - \sum_{i=0}^{1} u_i\right)\right)\right) \end{bmatrix} + \begin{bmatrix} L^{-1}\left(\frac{1}{S^{\beta}} L\left(N\left(\sum_{i=0}^{2} u_i\right) - N\left(\sum_{i=0}^{1} u_i\right)\right)\right) \\ L^{-1}\left(\frac{1}{S^{\beta}} L\left(N\left(\sum_{i=0}^{2} u_i\right) - N\left(\sum_{i=0}^{1} u_i\right)\right)\right) \end{bmatrix} \tag{19}$$

Continuing with this procedure, we get

$$\begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} u_{p+1} \\ v_{p+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L^{-1}\left(\frac{1}{S^{\beta}} L\left(\frac{\partial}{\partial x}\left(\sum_{i=0}^{p} u_i - \sum_{i=0}^{p-1} u_i\right)\right)\right) \\ L^{-1}\left(\frac{1}{S^{\beta}} L\left(\frac{\partial}{\partial x}\left(\sum_{i=0}^{p} u_i - \sum_{i=0}^{p-1} u_i\right)\right)\right) \end{bmatrix} + \begin{bmatrix} L^{-1}\left(\frac{1}{S^{\beta}} L\left(N\left(\sum_{i=0}^{p} u_i\right) - N\left(\sum_{i=0}^{p-1} u_i\right)\right)\right) \\ L^{-1}\left(\frac{1}{S^{\beta}} L\left(N\left(\sum_{i=0}^{p} u_i\right) - N\left(\sum_{i=0}^{p-1} u_i\right)\right)\right) \end{bmatrix}; p \in IN,\ p \geq 1,\ i = 0, 1, 2, \cdots;\ p+1 \tag{20}$$

Therefore the i^{th} term approximate solution vector of Equation $(8.a)$ given that Equation $(8.b)$ in power series form is given by

$$\begin{bmatrix} \widetilde{u}_i \\ \widetilde{v}_i \end{bmatrix} \cong \begin{bmatrix} u_0 + u_1 + u_2 + \cdots + u_{p+1} \\ v_0 + v_1 + v_2 + \cdots + v_{p+1} \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^{p+1} u_i \\ \sum_{i=0}^{p+1} v_i \end{bmatrix}; p = 1, 2, 3, \cdots \tag{21}$$

Step4. The infinite power series form solution vector of Equation $(8.a)$ given that Equation $(8.b)$ as $p \in IN$ approaches ∞, is obtained from Equation (21) and it is given as Equation (11).

Remark 4.1.1: If $Nu = 0$, then Equation $(8a)$ given that Equation $(8b)$ becomes Equation $(2.a)$ given that Equation $(2.b)$ and

i. u_0 which is given by Equation (16) becomes

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = L^{-1} \begin{bmatrix} L^{-1}\left(\frac{1}{S^{\beta}}\left(\sum_{r=0}^{p-1} u^{\,r} s^{\beta - r - 1}\right)\right) \\ L^{-1}\left(\frac{1}{S^{\beta}}\left(\sum_{r=0}^{p-1} v^{\,r} s^{\beta - r - 1}\right)\right) \end{bmatrix} \tag{22}$$

ii. u_1 which is given by Equation (17) becomes

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L^{-1}\left(\frac{1}{S^{\beta}} L\left(\frac{\partial}{\partial x}(u_0)\right)\right) \\ L^{-1}\left(\frac{1}{S^{\beta}} L\left(\frac{\partial}{\partial x}(v_0)\right)\right) \end{bmatrix} \tag{23}$$

iii. u_2 which is given by Equation (18) becomes

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L^{-1}\left(\frac{1}{S^\beta} L\left(\frac{\partial}{\partial x}\left(\sum_{i=0}^{1} u_i - u_0\right)\right)\right) \\ L^{-1}\left(\frac{1}{S^\beta} L\left(\frac{\partial}{\partial x}\left(\sum_{i=0}^{1} v_i - v_0\right)\right)\right) \end{bmatrix} \tag{24}$$

iv. u_3 which is given by Equation (19) becomes

$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_{2+1} \\ v_{2+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L^{-1}\left(\frac{1}{S^\beta} L\left(\frac{\partial}{\partial x}\left(\sum_{i=0}^{2} u_i - \sum_{i=0}^{1} u_i\right)\right)\right) \\ L^{-1}\left(\frac{1}{S^\beta} L\left(\frac{\partial}{\partial x}\left(\sum_{i=0}^{2} u_i - \sum_{i=0}^{1} u_i\right)\right)\right) \end{bmatrix} \tag{25}$$

v. Continuing with this procedure , $u_i = u_{p+1}$ which is given by Equation (20) becomes

$$\begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} u_{p+1} \\ v_{p+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L^{-1}\left(\frac{1}{S^\beta} L\left(\frac{\partial}{\partial x}\left(\sum_{i=0}^{p} u_i - \sum_{i=0}^{p-1} u_i\right)\right)\right) \\ L^{-1}\left(\frac{1}{S^\beta} L\left(\frac{\partial}{\partial x}\left(\sum_{i=0}^{p} u_i - \sum_{i=0}^{p-1} u_i\right)\right)\right) \end{bmatrix} ; p \in IN,\ p \geq 1,\ i = 0, 1, 2, \cdots, p+1 \tag{26}$$

4.2 Applications

The vectorial iterative fractional Laplace transform method (VIFLTM) was applied to four initial-value problems of fractional Cauchy-Riemann systems of the form Equation $(2.a)$ given that Equation $(2.b)$ for determining closed solution vectors in infinite fractional power series.

Example 1. Taking $f(x, y) = 0$ and $g(x, y) = 0$ in Equation $(2.a)$ and choosing $\phi(x) = 0$ and $\psi(x) = \sinh x$ in Equation $(2.b)$, consider the fractional Cauchy Riemann system

$$D_y^\beta \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}; 0 < \beta \leq 1, x \in IR, y > 0 \tag{27.a}$$

Subject to the initial conditions:

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix}; , x \in IR \tag{27.b}$$

By Equation (22),

$$\begin{bmatrix} u_0(x,y) \\ v_0(x,y) \end{bmatrix} = \begin{bmatrix} L^{-1}\left(\frac{1}{S^\beta}\left(u^0(x,0)s^{\beta-0-1}\right)\right) \\ L^{-1}\left(\frac{1}{S^\beta}\left(v^0(x,0)s^{\beta-0-1}\right)\right) \end{bmatrix} = \begin{bmatrix} L^{-1}\left(\frac{1}{S^\beta}\times 0\times s^{\beta-0-1}\right) \\ L^{-1}\left(\frac{1}{S^\beta}\left(\sinh x\times s^{\beta-0-1}\right)\right) \end{bmatrix} = \begin{bmatrix} L^{-1}(0) \\ L^{-1}\left(\frac{1}{S}(\sinh x)\right) \end{bmatrix} = \begin{bmatrix} 0 \\ \sinh x \end{bmatrix}$$

$$\begin{bmatrix} u_0(x,y) \\ v_0(x,y) \end{bmatrix} = \begin{bmatrix} 0 \\ \sinh x \end{bmatrix} \tag{28}$$

By Equation (23),

$$\begin{bmatrix} u_1(x,y) \\ v_1(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L^{-1}\left[\frac{1}{S^\beta}L\left[\left(\frac{\partial}{\partial x}(0)\right)\right]\right] \\ L^{-1}\left[\frac{1}{S^\beta}L\left[\left(\frac{\partial}{\partial x}(\sinh x)\right)\right]\right] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \cosh x\frac{y^\beta}{\Gamma(\beta+1)} \end{bmatrix} = \begin{bmatrix} \cosh x\frac{y^\beta}{\Gamma(\beta+1)} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_1(x,y) \\ v_1(x,y) \end{bmatrix} = \begin{bmatrix} \cosh x\dfrac{y^\beta}{\Gamma(\beta+1)} \\ 0 \end{bmatrix}, \quad 0<\beta\le 1, x\in IR, y>0 \tag{29}$$

By Equation (24),

$$\text{For } p=1, \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L^{-1}\left(\frac{1}{S^\beta}L\left(\frac{\partial}{\partial x}\left(\frac{(\cosh x)y^\beta}{\Gamma(\beta+1)}\right)\right)\right) \\ L^{-1}\left(\frac{1}{S^\beta}L\left(\frac{\partial}{\partial x}(0)\right)\right) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{(-\sinh x)y^{2\beta}}{\Gamma(2\beta+1)} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{(-\sinh x)y^{2\beta}}{\Gamma(2\beta+1)} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{(-\sinh x)y^{2\beta}}{\Gamma(2\beta+1)} \end{bmatrix}; 0<\beta\le 1, x\in \text{IR}, y>0 \tag{30}$$

By Equation (25),

$$\text{For } p=2, \begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L^{-1}\left(\frac{1}{S^\beta}L\left(\frac{\partial}{\partial x}(0)\right)\right) \\ L^{-1}\left(\frac{1}{S^\beta}L\left(\frac{\partial}{\partial x}\left(\frac{(-\sinh x)y^{2\beta}}{\Gamma(2\beta+1)}\right)\right)\right) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{(-\cosh x)y^{3\beta}}{\Gamma(3\beta+1)} \end{bmatrix} = \begin{bmatrix} \frac{(-\cosh x)y^{3\beta}}{\Gamma(3\beta+1)} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \dfrac{(-\cosh x)y^{3\beta}}{\Gamma(3\beta+1)} \\ 0 \end{bmatrix}; 0<\beta\le 1, x\in \text{IR}, y>0 \tag{31}$$

Continuing with this process, we obtain that

$$\begin{bmatrix} u_i(x,y) \\ v_i(x,y) \end{bmatrix} = \begin{bmatrix} u_{p+1}(x,y) \\ v_{p+1}(x,y) \end{bmatrix} = \begin{bmatrix} \cosh x \dfrac{(-1)^i y^{(2i+1)\beta}}{(\Gamma((2i+1)\beta+1)} \\ \sinh x \dfrac{(-1)^i y^{2i\beta}}{\Gamma((2i)\beta+1)} \end{bmatrix}, \ 0<\beta\le 1, \mathrm{x}\in I\mathrm{R}, y>0, i=1,\ 2,\ 3,\ \cdots, P+1, P\in IN \qquad (32)$$

The i^{th} order fractional power series solution vector of Equation $(27.a)$ given that Equation $(27.b)$, denoted by $\begin{bmatrix} \widetilde{u}_i(x,y) \\ \widetilde{v}_i(x,y) \end{bmatrix}$ is given by

$$\begin{bmatrix} \widetilde{u}_i(x,y) \\ \widetilde{v}_i(x,y) \end{bmatrix} = \begin{bmatrix} \widetilde{u}_{p+1}(x,y) \\ \widetilde{v}_{p+1}(x,y) \end{bmatrix} = \begin{bmatrix} \cosh x \sum_{i=0}^{p+1} \dfrac{(-1)^i y^{(2i+1)\beta}}{(\Gamma((2i+1)\beta+1)} \\ \sinh x \sum_{i=0}^{p+1} \dfrac{(-1)^i y^{2i\beta}}{\Gamma((2i)\beta+1)} \end{bmatrix}; \ 0<\beta\le 1, \mathrm{x}\in I\mathrm{R}, y>0, i=1,\ 2,\ 3,\ \cdots, P+1, P\in IN \quad (33)$$

By letting $p \in IN$ to ∞ or taking limit of both sides of Equation (33) as $p \in IN \to \infty$, the infinite fractional power series solution vector of Equation $(27.a)$ given that Equation $(27.b)$ denoted by $\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ is:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \cosh x \sum_{i=0}^{\infty} \dfrac{(-1)^i y^{(2i+1)\beta}}{(\Gamma((2i+1)\beta+1)} \\ \sinh x \sum_{i=0}^{\infty} \dfrac{(-1)^i y^{2i\beta}}{\Gamma((2i)\beta+1)} \end{bmatrix}; \ 0<\beta\le 1, \mathrm{x}\in I\mathrm{R}, y>0 \qquad (34)$$

Lastly, as β approaches to 1 from left, Equation (34) approaches to the exact solution vector of the ordinary (standard) Cauchy Riemann system of equations which can be obtained from Equation $(27.a)$ given that Equation $(27.b)$, and is given by

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \cosh x \sum_{i=0}^{\infty} \dfrac{(-1)^i y^{(2i+1)}}{(2i+1)!} \\ \sinh x \sum_{i=0}^{\infty} \dfrac{(-1)^i y^{2i}}{(2i)!} \end{bmatrix} = \begin{bmatrix} \cosh x \sin y \\ \sinh x \cos y \end{bmatrix}; \ 0<\beta\le 1, \mathrm{x}\in I\mathrm{R}, y>0 \qquad (35)$$

Example 2. Taking $f(x,y)=0$ and $g(x,y)=0$ in Equation $(2.a)$ and choosing $\phi(x)=\sin x$ and $\psi(x)=\cos x$ Equation $(2.b)$, consider the homogeneous fractional Cauchy Riemann system problem

$$\left\{ D_y^{\beta} \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}; 0 < \beta \le 1, x \in IR, y > 0 \right. \tag{36.a}$$

Subject to initial conditions:

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} \sin x \\ \cos x \end{bmatrix}; , x \in IR \tag{36.b}$$

By Equation (22),

$$\begin{bmatrix} u_0(x,y) \\ v_0(x,y) \end{bmatrix} = \begin{bmatrix} L^{-1}\left(\frac{1}{S^{\beta}} \left(u^0(x,0) s^{\beta-0-1} \right) \right) \\ L^{-1}\left(\frac{1}{S^{\beta}} \left(v^0(x,0) s^{\beta-0-1} \right) \right) \end{bmatrix} = \begin{bmatrix} L^{-1}\left(\frac{1}{S^{\beta}} \times \sin x \times s^{\beta-0-1} \right) \\ L^{-1}\left(\frac{1}{S^{\beta}} \left(\cos x \times s^{\beta-0-1} \right) \right) \end{bmatrix} = \begin{bmatrix} L^{-1}\left(\frac{1}{S} \left(\sin x \right) \right) \\ L^{-1}\left(\frac{1}{S} \left(\cos x \right) \right) \end{bmatrix} = \begin{bmatrix} \sin x \\ \cos x \end{bmatrix}$$

$$\begin{bmatrix} u_0(x,y) \\ v_0(x,y) \end{bmatrix} = \begin{bmatrix} \sin x \\ \cos x \end{bmatrix} \tag{37}$$

By Equation (23),

$$\begin{bmatrix} u_1(x,y) \\ v_1(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L^{-1}\left[\frac{1}{S^{\beta}} L\left[\left(\frac{\partial}{\partial x} (\sin x) \right) \right] \right] \\ L^{-1}\left[\frac{1}{S^{\beta}} L\left[\left(\frac{\partial}{\partial x} (\cos x) \right) \right] \right] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{(\cos x) y^{\beta}}{\Gamma(\beta+1)} \\ \frac{(-\sin x) y^{\beta}}{\Gamma(\beta+1)} \end{bmatrix} = \begin{bmatrix} \frac{(-\sin x) y^{\beta}}{\Gamma(\beta+1)} \\ \frac{(-\cos x) y^{\beta}}{\Gamma(\beta+1)} \end{bmatrix}$$

$$\begin{bmatrix} u_1(x,y) \\ v_1(x,y) \end{bmatrix} = \begin{bmatrix} \frac{(-\sin x) y^{\beta}}{\Gamma(\beta+1)} \\ \frac{(-\cos x) y^{\beta}}{\Gamma(\beta+1)} \end{bmatrix}, \quad 0 < \beta \le 1, x \in IR, y > 0 \tag{38}$$

By Equation (24):

For $p = 1$, $\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L^{-1}\left(\frac{1}{S^{\beta}} L\left(\frac{\partial}{\partial x} \left(\frac{(-\sin x) y^{\beta}}{\Gamma(\beta+1)} \right) \right) \right) \\ L^{-1}\left(\frac{1}{S^{\beta}} L\left(\frac{\partial}{\partial x} \left(\frac{(-\cos x) y^{\beta}}{\Gamma(\beta+1)} \right) \right) \right) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{(-\cos x) y^{2\beta}}{\Gamma(2\beta+1)} \\ \frac{(\sin x) y^{2\beta}}{\Gamma(2\beta+1)} \end{bmatrix} = \begin{bmatrix} \frac{(\sin x) y^{2\beta}}{\Gamma(2\beta+1)} \\ \frac{(\cos x) y^{2\beta}}{\Gamma(2\beta+1)} \end{bmatrix}$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{(\sin x) y^{2\beta}}{\Gamma(2\beta+1)} \\ \frac{(\cos x) y^{2\beta}}{\Gamma(2\beta+1)} \end{bmatrix}; 0 < \beta \le 1, x \in \mathrm{IR}, y > 0 \tag{39}$$

By Equation (25),

For $p=2$, $\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L^{-1}\left(\frac{1}{S^\beta} L\left(\frac{\partial}{\partial x}\left(\frac{(\sin x)y^{2\beta}}{\Gamma(2\beta+1)}\right)\right)\right) \\ L^{-1}\left(\frac{1}{S^\beta} L\left(\frac{\partial}{\partial x}\left(\frac{(\cos x)y^{2\beta}}{\Gamma(2\beta+1)}\right)\right)\right) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{(\cos x)y^{3\beta}}{\Gamma(3\beta+1)} \\ \frac{(-\sin x)y^{3\beta}}{\Gamma(3\beta+1)} \end{bmatrix} = \begin{bmatrix} \frac{(-\sin x)y^{3\beta}}{\Gamma(3\beta+1)} \\ \frac{(-\cos x)y^{3\beta}}{\Gamma(3\beta+1)} \end{bmatrix}$

$$\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{(-\sin x)y^{3\beta}}{\Gamma(3\beta+1)} \\ \frac{(-\cos x)y^{3\beta}}{\Gamma(3\beta+1)} \end{bmatrix}; 0 < \beta \le 1, x \in \text{IR}, y > 0 \tag{40}$$

Continuing with this process, we obtain that

$$\begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} u_{p+1} \\ v_{p+1} \end{bmatrix} = \begin{bmatrix} \sin x \frac{(-1)^i y^{i\beta}}{(\Gamma(i\beta+1)} \\ \cos x \frac{(-1)^i y^{i\beta}}{\Gamma(i\beta+1)} \end{bmatrix}, \quad 0 < \beta \le 1, \text{x} \in I\text{R}, y > 0, i = 1,\ 2,\ 3,\ \cdots, P+1, P \in IN \tag{41}$$

The i^{th} order approximate fractional power series form solution vector of Equation $(36.a)$ given that Equation $(36.b)$, denoted by $\begin{bmatrix} \widetilde{u}_i(x,y) \\ \widetilde{v}_i(x,y) \end{bmatrix}$ is given by:

$$\begin{bmatrix} \widetilde{u}_i(x,y) \\ \widetilde{v}_i(x,y) \end{bmatrix} = \begin{bmatrix} \widetilde{u}_{p+1}(x,y) \\ \widetilde{v}_{p+1}(x,y) \end{bmatrix} = \begin{bmatrix} \sin x \sum_{i=0}^{p+1} \frac{(-1)^i y^{i\beta}}{(\Gamma(i\beta+1)} \\ \cos x \sum_{i=0}^{p+1} \frac{(-1)^i y^{i\beta}}{\Gamma(i\beta+1)} \end{bmatrix}; \ 0 < \beta \le 1, \text{x} \in I\text{R}, y > 0, i = 1,\ 2,\ 3,\ \cdots, P+1, P \in IN \tag{42}$$

By letting $p \in IN$ to ∞ or taking limit of both sides of Equation (42) as $p \in IN \to \infty$, the of infinite fractional power series form solution vector of Equation $(36.a)$ given that Equation $(36.b)$ denoted by $\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ is

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \sin x \sum_{i=0}^{\infty} \frac{(-1)^i y^{i\beta}}{(\Gamma(i\beta+1)} \\ \cos x \sum_{i=0}^{\infty} \frac{(-1)^i y^{i\beta}}{\Gamma(i\beta+1)} \end{bmatrix}; \ 0 < \beta \le 1, \text{x} \in I\text{R}, y > 0 \tag{43}$$

Lastly, as β approaches to 1 from left, Equation (43) approaches to the exact solution vector of the ordinary (standard) Cauchy Riemann system of equations, which can be obtained from Equation $(36.a)$ given that Equation $(36.b)$, and is given by

$$\begin{bmatrix} u_{exact}(x,y) \\ v_{exact}(x,y) \end{bmatrix} = \begin{bmatrix} \sin x \sum_{i=0}^{\infty} \frac{(-y)^i}{i!} \\ \cos x \sum_{i=0}^{\infty} \frac{(-y)^i}{i!} \end{bmatrix} = \begin{bmatrix} e^{-y} \sin x \\ e^{-y} \cos x \end{bmatrix}; \ 0 < \beta \le 1, \mathrm{x} \in I\mathrm{R}, y > 0 \tag{44}$$

Example 3.Taking $f(x,y) = y \sin ax$ and $g(x,y) = 0$ in Equation $(2.a)$ and choosing $\phi(x) = \sin ax$ and $\psi(x) = \cos ax$ in Equation $(2.b)$, consider the following inhomogeneous fractional Cauchy Riemann system problem

$$D_y^{\beta} \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} + \begin{bmatrix} y \sin ax \\ 0 \end{bmatrix}; 0 < \beta \le 1, x \in IR, y > 0 \tag{45.a}$$

Subject to initial conditions:

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} \sin ax \\ \cos ax \end{bmatrix}; , x \in IR \tag{45.b}$$

By Equation (22),

$$\begin{bmatrix} u_0(x,y) \\ v_0(x,y) \end{bmatrix} = \begin{bmatrix} L^{-1}\left(\frac{1}{S^{\beta}}\left(u^0(x,0)s^{\beta-0-1} + Lf(x,y)\right)\right) \\ L^{-1}\left(\frac{1}{S^{\beta}}\left(v^0(x,0)s^{\beta-0-1}\right) + Lg(x,y)\right) \end{bmatrix} = \begin{bmatrix} L^{-1}\left(\frac{\sin ax \times s^{\beta-0-1}}{s^{\beta}} + \frac{L(y \sin ax)}{s^{\beta}}\right) \\ L^{-1}\left(\frac{1}{S^{\beta}}\left(\cos ax \times s^{\beta-0-1}\right)\right) \end{bmatrix} = \begin{bmatrix} L^{-1}\left(\frac{\sin ax}{s} + \frac{1}{s^{\beta}} \frac{\sin ax}{s^2}\right) \\ L^{-1}\left(\frac{\cos ax}{s}\right) \end{bmatrix}$$

$$\begin{bmatrix} u_0(x,y) \\ v_0(x,y) \end{bmatrix} = \begin{bmatrix} \sin ax + \frac{y^{\beta+1}}{\Gamma(\beta+2)} \sin ax \\ \cos ax \end{bmatrix} \tag{46}$$

By Equation (23),

$$\begin{bmatrix} u_1(x,y) \\ v_1(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L^{-1}\left[\frac{1}{S^{\beta}} L\left[\left(\frac{\partial}{\partial x}(\sin ax + \frac{y^{\beta+1}}{\Gamma(\beta+2)} \sin ax)\right)\right]\right] \\ L^{-1}\left[\frac{1}{S^{\beta}} L\left[\left(\frac{\partial}{\partial x}(\cos ax)\right)\right]\right] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{(a \cos ax) y^{\beta 1}}{\Gamma(\beta+1)} + \frac{(a \cos ax) y^{2\beta+1}}{\Gamma(2\beta+2)} \\ \frac{-a\, y^{\beta} \sin ax}{\Gamma(\beta+1)} \end{bmatrix} = \begin{bmatrix} \frac{-a\, y^{\beta} \sin ax}{\Gamma(\beta+1)} \\ -\frac{(a \cos ax) y^{\beta}}{\Gamma(\beta+1)} - \frac{(a \cos ax) y^{2\beta+1}}{\Gamma(2\beta+2)} \end{bmatrix}$$

$$\begin{bmatrix} u_1(x,y) \\ v_1(x,y) \end{bmatrix} = \begin{bmatrix} \frac{-a y^{\beta} \sin ax}{\Gamma(\beta+1)} \\ -\frac{(a \cos ax) y^{\beta}}{\Gamma(\beta+1)} - \frac{(a \cos ax) y^{2\beta+1}}{\Gamma(2\beta+2)} \end{bmatrix}, \ 0 < \beta \le 1, x \in IR, y > 0 \tag{47}$$

By Equation (24):

For $p = 1$,

$$\begin{bmatrix} u_2(x,y) \\ v_2(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L^{-1}\left[\frac{1}{S^{\beta}} L\left[\left(\frac{\partial}{\partial x}\left(\frac{-a\, y^{\beta} \sin ax}{\Gamma(\beta+1)}\right)\right)\right]\right] \\ L^{-1}\left[\frac{1}{S^{\beta}} L\left[\left(\frac{\partial}{\partial x}\left(-\frac{(a \cos ax) y^{\beta}}{\Gamma(\beta+1)} - \frac{(a \cos ax) y^{2\beta+1}}{\Gamma(2\beta+2)}\right)\right)\right]\right] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} L^{-1} \begin{bmatrix} \frac{-a^2 \cos ax}{s^{2\beta+1}} \\ \frac{(a^2 \sin ax) y^{2\beta}}{\Gamma(2\beta+1)} + \frac{(a^2 \sin ax) y^{3\beta+1}}{\Gamma(3\beta+2)} \end{bmatrix} = \begin{bmatrix} \frac{(a^2 \sin ax) y^{2\beta}}{\Gamma(2\beta+1)} + \frac{(a^2 \sin ax) y^{3\beta+1}}{\Gamma(3\beta+2)} \\ \frac{(a^2 \cos ax) y^{2\beta}}{\Gamma(2\beta+1)} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{\left(a^2 \sin ax\right)y^{2\beta}}{\Gamma(2\beta+1)} + \frac{\left(a^2 \sin ax\right)y^{3\beta+1}}{\Gamma(3\beta+2)} \\ \frac{\left(a^2 \cos ax\right)y^{2\beta}}{\Gamma(2\beta+1)} \end{bmatrix}; 0 < \beta \le 1, x \in \text{IR}, y > 0 \quad (48)$$

By Equation (25),

For $p = 2$,

$$\begin{bmatrix} u_3(x,y) \\ v_3(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L^{-1}\left[\frac{1}{S^\beta} L\left[\left(\frac{\partial}{\partial x}\left(\frac{\left(a^2 \sin ax\right)y^{2\beta}}{\Gamma(2\beta+1)} + \frac{\left(a^2 \sin ax\right)y^{3\beta+1}}{\Gamma(3\beta+2)}\right)\right)\right]\right] \\ L^{-1}\left[\frac{1}{S^\beta} L\left[\left(\frac{\partial}{\partial x}\frac{\left(a^2 \cos ax\right)y^{2\beta}}{\Gamma(2\beta+1)}\right)\right]\right] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\left(a^3 \cos ax\right)y^{3\beta}}{\Gamma(3\beta+1)} + \frac{\left(a^3 \cos ax\right)y^{4\beta+1}}{\Gamma(4\beta+2)} \\ \frac{-a^3 y^{3\beta} \sin ax}{\Gamma(3\beta+1)} \end{bmatrix} = \begin{bmatrix} \frac{-a^3 y^{3\beta} \sin ax}{\Gamma(3\beta+1)} \\ -\frac{\left(a^3 \cos ax\right)y^{3\beta}}{\Gamma(3\beta+1)} - \frac{\left(a^3 \cos ax\right)y^{4\beta+1}}{\Gamma(4\beta+2)} \end{bmatrix}$$

$$\begin{bmatrix} u_3(x,y) \\ v_3(x,y) \end{bmatrix} = \begin{bmatrix} \frac{-a^3 y^{3\beta} \sin ax}{\Gamma(3\beta+1)} \\ -\frac{\left(a^3 \cos ax\right)y^{3\beta}}{\Gamma(3\beta+1)} - \frac{\left(a^3 \cos ax\right)y^{4\beta+1}}{\Gamma(4\beta+2)} \end{bmatrix}, \quad 0 < \beta \le 1, x \in IR, y > 0 \quad (49)$$

Continuing with this process, we obtain that

$$\begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} u_{p+1} \\ v_{p+1} \end{bmatrix} = \begin{bmatrix} \sin ax\left(\frac{(-a)^i y^{i\beta}}{(\Gamma(i\beta+1)} + \frac{a^{2i} y^{(2i+1)\beta+1}}{\Gamma((2i+1)\beta+2)}\right) \\ \cos ax\left(\frac{(-a)^i y^{i\beta}}{\Gamma(i\beta+1)} - \frac{a^{2i+1} y^{(2i+2)\beta+1}}{\Gamma((2i+2)\beta+2)}\right) \end{bmatrix}, \quad 0 < \beta \le 1, \text{x} \in IR, y > 0, i = 1,\ 2,\ 3,\ \cdots, P+1, P \in IN \quad (50)$$

The i^{th} order approximate fractional power series form solution vector of Equation $(45.a)$ given that Equation $(45.b)$, denoted by $\begin{bmatrix} \tilde{u}_i(x,y) \\ \tilde{v}_i(x,y) \end{bmatrix}$ is given by:

$$\begin{bmatrix} \tilde{u}_i(x,y) \\ \tilde{v}_i(x,y) \end{bmatrix} = \begin{bmatrix} \tilde{u}_{p+1}(x,y) \\ \tilde{v}_{p+1}(x,y) \end{bmatrix} = \begin{bmatrix} \sin ax \sum_{i=0}^{p+1}\left(\frac{(-a)^i y^{i\beta}}{(\Gamma(i\beta+1)} + \frac{a^{2i} y^{(2i+1)\beta+1}}{\Gamma((2i+1)\beta+2)}\right) \\ \cos ax \sum_{i=0}^{p+1}\left(\frac{(-a)^i y^{i\beta}}{\Gamma(i\beta+1)} - \frac{a^{2i+1} y^{(2i+2)\beta+1}}{\Gamma((2i+2)\beta+2)}\right) \end{bmatrix}; \ 0 < \beta \le 1, \text{x} \in IR, y > 0, i = 1,2,3,\cdots; P+1, P \in IN \quad (51)$$

By letting $p \in IN$ to ∞ or taking limit of both sides of Equation (51) as $p \in IN \to \infty$, the solution vector of Equation $(45.a)$ given that Equation $(45.b)$ in the form of infinite fractional power series denoted by $\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ is:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \sin ax \sum_{i=0}^{\infty}\left(\frac{(-a)^i y^{i\beta}}{(\Gamma(i\beta+1)} + \frac{a^{2i} y^{(2i+1)\beta+1}}{\Gamma((2i+1)\beta+2)}\right) \\ \cos ax \sum_{i=0}^{\infty}\left(\frac{(-a)^i y^{i\beta}}{\Gamma(i\beta+1)} - \frac{a^{2i+1} y^{(2i+2)\beta+1}}{\Gamma((2i+2)\beta+2)}\right) \end{bmatrix}; \ 0 < \beta \le 1, \text{x} \in IR, y > 0 \quad (52)$$

Lastly, as β approaches to 1 from left, Equation (52) approaches to the exact solution vector of the ordinary (standard) Cauchy Riemann system of equations, which can be obtained from Equation $(45.a)$ given that Equation $(45.b)$ and is given by

$$\begin{bmatrix} u_{exact}(x,y) \\ v_{exact}(x,y) \end{bmatrix} = \begin{bmatrix} \sin ax \sum_{i=0}^{\infty} \left(\frac{(-ay)^i}{i!} + \frac{(ay)^{2i} y^2}{(2i+2)!} \right) \\ \cos ax \sum_{i=0}^{\infty} \left(\frac{(-ay)^i}{i!} - \frac{a(ay)^{2i} y^3}{(2i+3)!} \right) \end{bmatrix}; \ 0 < \beta \le 1, \mathrm{x} \in I\mathrm{R}, y > 0 \tag{53}$$

Example 4. Taking $f(x,y) = xy$ and $g(x,y) = xy$ in Equation $(2.a)$ and choosing $\phi(x) = 0$ and $\psi(x) = 0$ Equation $(2.b)$, consider the following inhomogeneous fractional Cauchy Riemann system problem

$$D_y^{\beta} \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} + \begin{bmatrix} xy \\ xy \end{bmatrix}; 0 < \beta \le 1, x \in IR, y > 0 \tag{54.a}$$

Subject to initial conditions:

$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; , x \in IR \tag{54.b}$$

By Equation (22),

$$\begin{bmatrix} u_0(x,y) \\ v_0(x,y) \end{bmatrix} = \begin{bmatrix} L^{-1}\left(\frac{1}{S^{\beta}} \left(u^0(x,0) s^{\beta-0-1} + Lf(x,y) \right) \right) \\ L^{-1}\left(\frac{1}{S^{\beta}} \left(v^0(x,0) s^{\beta-0-1} \right) + Lg(x,y) \right) \end{bmatrix} = \begin{bmatrix} L^{-1}\left(\frac{0 \times s^{\beta-0-1}}{s^{\beta}} + \frac{L(e^x y)}{s^{\beta}} \right) \\ L^{-1}\left(\frac{1}{S^{\beta}} \left(0 \times s^{\beta-0-1} \right) + \frac{L(e^x y)}{s^{\beta}} \right) \end{bmatrix} = \begin{bmatrix} L^{-1}\left(\frac{xy}{s^{\beta} \times s^2} \right) \\ L^{-1}\left(\frac{xy}{s^{\beta} \times s^2} \right) \end{bmatrix}$$

$$\begin{bmatrix} u_0(x,y) \\ v_0(x,y) \end{bmatrix} = \begin{bmatrix} \frac{xy^{\beta+1}}{\Gamma(\beta+2)} \\ \frac{xy^{\beta+1}}{\Gamma(\beta+2)} \end{bmatrix} \tag{55}$$

By Equation (23),

$$\begin{bmatrix} u_1(x,y) \\ v_1(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L^{-1}\left[\frac{1}{S^{\beta}} L\left[\left(\frac{\partial}{\partial x} \left(\frac{xy^{\beta+1}}{\Gamma(\beta+2)} \right) \right) \right] \right] \\ L^{-1}\left[\frac{1}{S^{\beta}} L\left[\left(\frac{\partial}{\partial x} \left(\frac{xy^{\beta+1}}{\Gamma(\beta+2)} \right) \right) \right] \right] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{y^{2\beta+1}}{\Gamma(2\beta+2)} \\ \frac{y^{\beta+1}}{\Gamma(2\beta+1)} \end{bmatrix} = \begin{bmatrix} \frac{y^{2\beta+1}}{\Gamma(2\beta+2)} \\ -\frac{y^{2\beta+1}}{\Gamma(2\beta+2)} \end{bmatrix}$$

$$\begin{bmatrix} u_1(x,y) \\ v_1(x,y) \end{bmatrix} = \begin{bmatrix} \frac{y^{2\beta+1}}{\Gamma(2\beta+2)} \\ -\frac{y^{2\beta+1}}{\Gamma(2\beta+2)} \end{bmatrix}, \ 0 < \beta \le 1, x \in IR, y > 0 \tag{56}$$

By Equation (24):

For $p=1$, $\begin{bmatrix} u_2(x,y) \\ v_2(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L^{-1}\left[\frac{1}{S^\beta} L\left[\left(\frac{\partial}{\partial x}\left(\frac{y^{2\beta+1}}{\Gamma(2\beta+2)}\right)\right)\right]\right] \\ L^{-1}\left[\frac{1}{S^\beta} L\left[\left(\frac{\partial}{\partial x}\left(-\frac{y^{2\beta+1}}{\Gamma(2\beta+2)}\right)\right)\right]\right] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; 0 < \beta \le 1, x \in \mathrm{IR}, y > 0 \tag{57}$$

By Equation (25),

For $p=2$, $\begin{bmatrix} u_3(x,y) \\ v_3(x,y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} L^{-1}\left[\frac{1}{S^\beta} L\left[\left(\frac{\partial}{\partial x}(0)\right)\right]\right] \\ L^{-1}\left[\frac{1}{S^\beta} L\left[\left(\frac{\partial}{\partial x}(0)\right)\right]\right] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} u_3(x,y) \\ v_3(x,y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad 0 < \beta \le 1, \ x \in IR, \ y > 0 \tag{58}$$

Continuing with this process, we obtain that

$$\begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} u_{p+1} \\ v_{p+1} \end{bmatrix} = \begin{bmatrix} \frac{xy^{\beta+1}}{\Gamma(\beta+2)} + \frac{y^{2\beta+1}}{\Gamma(2\beta+2)} \\ \frac{xy^{\beta+1}}{\Gamma(\beta+2)} - \frac{y^{2\beta+1}}{\Gamma(2\beta+2)} \end{bmatrix}, \quad 0 < \beta \le 1, \mathrm{x} \in I\mathrm{R}, y > 0, i = 1, 2, 3, \cdots, P+1, P \in IN \tag{59}$$

The i^{th} order approximate fractional power series form solution vector of Equation $(54.a)$ given that Equation $(54.b)$, denoted by $\begin{bmatrix} \tilde{u}_i(x,y) \\ \tilde{v}_i(x,y) \end{bmatrix}$ is given by:

$$\begin{bmatrix} \tilde{u}_i(x,y) \\ \tilde{v}_i(x,y) \end{bmatrix} = \begin{bmatrix} \tilde{u}_{p+1}(x,y) \\ \tilde{v}_{p+1}(x,y) \end{bmatrix} = \begin{bmatrix} \frac{xy^{\beta+1}}{\Gamma(\beta+2)} + \frac{y^{2\beta+1}}{\Gamma(2\beta+2)} \\ \frac{xy^{\beta+1}}{\Gamma(\beta+2)} - \frac{y^{2\beta+1}}{\Gamma(2\beta+2)} \end{bmatrix}; \ 0 < \beta \le 1, \ \mathrm{x} \in I\mathrm{R}, y > 0, i = 1, 2, 3, \cdots, P+1, \ P \in IN \tag{60}$$

By letting $p \in IN$ to ∞ or taking limit of both sides of Equation (60) as $p \in IN \to \infty$, the solution vector of Equation $(54.a)$ given that Equation $(54.b)$ in the form of infinite fractional power series denoted by $\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$ is:

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} \frac{xy^{\beta+1}}{\Gamma(\beta+2)} + \frac{y^{2\beta+1}}{\Gamma(2\beta+2)} \\ \frac{xy^{\beta+1}}{\Gamma(\beta+2)} - \frac{y^{2\beta+1}}{\Gamma(2\beta+2)} \end{bmatrix}; \quad 0 < \beta \le 1, \ \mathrm{x} \in I\mathrm{R}, y > 0 \tag{61}$$

Lastly, as β approaches to 1 from left, Equation (61) approaches to the exact solution vector of the ordinary (standard) Cauchy Riemann system of equations, which can be obtained from Equation $(54.a)$ given that equation $(54.b)$, and is given by

$$\begin{bmatrix} u_{exact}(x,y) \\ v_{exact}(x,y) \end{bmatrix} = \begin{bmatrix} \frac{xy^2}{2} + \frac{y^3}{3!} \\ \frac{xy^2}{2} - \frac{y^3}{3!} \end{bmatrix}; \quad 0 < \beta \leq 1,\ x \in R,\ y > 0 \tag{62}$$

4.3 Discussion

Through the four examples considered above, the vectorial iterative fractional Laplace transform method (VIFLTM) was successfully applied to initial-value problems fractional Cauchy-Riemann systems of the form Equation $(2.a)$ given that Equation $(2.b)$ for: $f(x,y)=0$ and $g(x,y)=0$ with initial conditions $\phi(x)=0$ and $\psi(x)=\sinh x$; $f(x,y)=0$ and $g(x,y)=0$ with initial conditions $\phi(x)=\sin x$ and $\psi(x)=\cos x$; $f(x,y)=y\sin ax$ and $g(x,y)=0$ with initial conditions $\phi(x)=\sin ax$ and $\psi(x)=\cos ax$; $f(x,y)=xy$ and $g(x,y)=xy$ with initial conditions $\phi(x)=0$ and $\psi(x)=0$ for $0<\beta\leq 1$.

Through examples one and two, the solution vectors of Equation $(2.a)$ given that Equation $(2.b)$ in the form of infinite fractional power series was obtained and the solution vectors are in complete agreement with the results of Naseem and Tahir [48] for $\beta=1$. So, the solution vectors of Equation $(2.a)$ given that Equation $(2.b)$ in the form of infinite fractional power series generalizes the results in Naseem and Tahir [48].

Applying VIFLTM to Equation $(2.a)$ given that Equation $(2.b)$ through the second and third examples where $f(x,y)=y\sin ax$ and $g(x,y)=0$ with initial conditions $\phi(x)=\sin ax$ and $\psi(x)=\cos ax$ for $0<\beta\leq 1$; $f(x,y)=y\sin ax$ and $g(x,y)=0$ with initial conditions $\phi(x)=\sin ax$ and $\psi(x)=\cos ax$ for $0<\beta\leq 1$; $f(x,y)=xy$ and $g(x,y)=xy$ with initial conditions $\phi(x)=0$ and $\psi(x)=0$ for $0<\beta\leq 1$, the solution vectors in the form of infinite fractional power series were obtained successfully.

5 Conclusion

In this paper, basic idea of vectorial iterative fractional Laplace transform method (VIFLTM) for solving fractional Cauchy-Riemann System equations with initial conditions with initial conditions of the form $(2.a)$ given that Equation $(2.b)$ was developed and it was successfully applied to fractional Cauchy-Riemann System equations with initial conditions to obtain their closed solution vectors in the form of infinite fractional power series with a minimum size of calculations.

Thus, we can conclude that the VIFLTM used in solving fractional Cauchy-Riemann System equations with initial conditions can be extended to solve other fractional partial differential equations with initial conditions which can arise in fields of sciences.

Competing Interests

Author has declared that no competing interests exist.

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