Some Fixed Point Theorems in S-metric Spaces via Simulation Function

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

Original Research Article

Abstract

We introduce the concept of generalized β-γ-Z contraction mapping with respect to a simulation function ξ and study the existence of fixed points for such mappings in complete S-metric spaces. Further, we extend it to partially ordered complete S-metric spaces.

Keywords: $\beta - \gamma - Z$ contraction; fixed point; S-metric space.

Mathematics Subject Classification: 47H10, 54H25.

1 Introduction

Fixed point theory is one of the most effective and fruitful tool in mathematics which has very enormous applications within as well as outside mathematics. Starting of fixed point theory is from the celebrated Banach Contraction Principle [1], many authors have obtained its several generalizations in different ways. The famous

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Banach contraction principle, established by Banach guarantees the presence and uniqueness of fixed points in complete metric spaces. A metrical common fixed point theorem generally involves conditions on continuity, commutativity, and contraction of the given mapping, as well as completeness or closedness of the space, along with conditions on suitable containment amongst the range of mappings [2-6]. By introducing various contractions in various ambient spaces, several researchers generalized and extended this theory (See [7-14]).

Definition 1.1. [15]

Let X be a non-empty set, a S-metric on X is a function $S : X^3 \rightarrow [0, +\infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,

1. $S(x, y, z) \geq 0$,
2. $S(x, y, z) = 0$ if and only if $x = y = z$,
3. $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$,

for all $x, y, z, a \in X$.

The pair (X, S) is called an S-metric space.

Definition 1.2. [16]

Let (X, S) be an S-metric space.

(i) A sequence $\{x_n\} \subset X$ converges to $x \in X$ if $S(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow +\infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $S(x_n, x_n, x) < \epsilon$.

(ii) A sequence $\{x_n\} \subset X$ is a Cauchy sequence if $S(x_n, x_n, x_m) \rightarrow 0$ as $n, m \rightarrow +\infty$. That is for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $S(x_n, x_n, x_m) < \epsilon$.

(iii) S-metric space (X, S) is complete if every Cauchy sequence is a convergent sequence.

Lemma 1.3. [16] Suppose (X, S) is a metric space. Let us consider a sequence $\{x_n\}$ in X such that $S(x_n, x_n, x_{n+1}) \rightarrow 0$ as $n \rightarrow \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exists an $\epsilon > 0$ and sequences of positive integers $\{r_k\}$ and $\{s_k\}$ with $s_k > r_k > k$ such that $S(x_{r_k}, x_{r_k}, x_{s_k}) \geq \epsilon$. For each $k > 0$, corresponding to r_k, we can choose s_k to be the smallest positive integer such that $S(x_{r_k}, x_{r_k}, x_{s_k}) \geq \epsilon, S(x_{r_k}, x_{r_k}, x_{s_{k-1}}) < \epsilon$ and

i. $\lim_{k \to \infty} S(x_{s_{k-1}}, x_{s_{k-1}}, x_{r_{k+1}}) = \epsilon$

ii. $\lim_{k \to \infty} S(x_{s_k}, x_{s_k}, x_{r_k}) = \epsilon$

iii. $\lim_{k \to \infty} S(x_{r_{k-1}}, x_{r_{k-1}}, x_{s_k}) = \epsilon$

iv. $\lim_{k \to \infty} S(x_{s_k}, x_{s_k}, x_{r_{k+1}}) = \epsilon$.

Definition 1.4. [17] A simulation function is a mapping

$$\xi : [0, \infty) \times [0, \infty) \rightarrow (-\infty, \infty)$$

Satisfying the following conditions:

1) $\xi(0, 0) = 0$;
2) $\xi(g, h) < g - h$, for all $g, h > 0$;
3) If $\{g_n\}, \{h_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n \to \infty} g_n = \lim_{n \to \infty} h_n = m \in (0, \infty)$, then $\lim_{n \to \infty} \sup \xi(g_n, h_n) < 0$.

Remark 1.5. [17] Let ξ be a simulation function, if $\{g_n\}, \{h_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n \to \infty} g_n = \lim_{n \to \infty} h_n = m \in (0, \infty)$, then $\lim_{n \to \infty} \sup \xi(k g_n, h_n) < 0$ for any $k > 1$.

An example of simulation function is as follows:

Example 1.6. [17] Let $\xi : [0, \infty) \times [0, \infty) \to (-\infty, \infty)$, be defined by

1) $\xi(g, h) = \delta h - g$ for all $g, h \in [0, \infty)$, where $\delta \in [0, 1)$.
2) $\xi(h, g) = \frac{h}{1+h} - g$ for all $g, h \in [0, \infty)$.
3) $\xi(g, h) = h - kg$ otherwise, where $k > 1$.
4) $\xi(h, g) = \frac{h}{1+h} - ge^g$ for all $g, h \in [0, \infty)$.

2 Fixed Point Theorems for Generalized $\beta - \gamma - Z$ Contraction

Here we prove some fixed point theorems for generalized $\beta - \gamma - Z$ contraction with respect to ξ.

Definition 2.1. Let $(X, \mathcal{S})$ be a $\mathcal{S}$-metric space and T be a self-map on X. Let $\beta, \gamma : X \times X \times X \to [0, \infty)$ be two functions. We say that T is β-admissible mapping with respect to γ if $x, y, z \in X$ then $\beta(x, y, z) \geq \gamma(x, y, z)$ implies that $\beta(Tx, Ty, Tz) \geq \gamma(Tx, Ty, Tz)$.

Definition 2.2. Let $(X, \mathcal{S})$ be a S-metric space and $\beta, \gamma : X \times X \times X \to [0, \infty)$. A mapping $T : X \to X$ is said to be β-γ-continuous if every sequence $\{x_n\}$ in X with $\beta(x_n, x_n, x_{n+1}) \geq \gamma(x_n, x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$ implies $Tx_n \to Tx$ as $n \to \infty$.

Definition 2.3. Let $(X, \mathcal{S})$ be a S-metric space and $\beta, \gamma : X \times X \times X \to [0, \infty)$ be two functions. A mapping $T : X \to X$ is said to be a generalized $\beta - \gamma - Z$ contraction with respect to ξ If there exists simulation function ξ such that

$$\xi(\mathcal{S}(Tx, Tx, Ty), M_T(x, x, y)) \geq 0 \text{ for any } x, y \in X, \tag{2.1}$$

with $\beta(x, x, y) \geq \gamma(x, x, y)$,

where $M_T(x, x, y) = max\{\mathcal{S}(x, x, y), \mathcal{S}(x, x, Tx), \mathcal{S}(y, y, Ty), \frac{\mathcal{S}(x,x,Ty) + \mathcal{S}(y,y,Tx)}{2}\}$.

Theorem 2.4. Let $(X, \mathcal{S})$ be a S-metric space, $T : X \to X$ and $\beta, \gamma : X \times X \times X \to [0, \infty)$ are mappings.

Suppose that the following conditions are satisfied:

    I.   is a generalized β-γ- Z contraction with respect to ξ.
    II. T is a β-admissible mapping with respect to γ,
    III.   there exists $x_0 \in X$ such that $\beta(x_0, x_0, Tx_0) \geq \gamma(x_0, x_0, Tx_0)$ and
    IV. T is an β-γ-continuous mapping.

Then T has a fixed point $q \in X$ and $\{T^n x_0\}$ converges to q.

Proof. Let $x_0 \in X$ be as in (iii), so that $\beta(x_0, x_0, Tx_0) \geq \gamma(x_0, x_0, Tx_0)$. Now we define a sequence $\{x_h\}$ in X by $x_{h+1} = T^h x_1 = Tx_h$ for all $h \in \mathbb{N}$. Suppose that $x_{h_0} = x_{h_{0+1}}$ for some $h_0 \in \mathbb{N}$, we have $Tx_{h_0} = x_{h_0}$, so that x_{h_0} is a fixed point of T.

Hence, without loss of generality, we assume that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$.

From (2.1), we have

$$\xi(\mathcal{S}(x_{h+1}, x_{h+1}, x_{h+2}), M_T(x_h, x_h, x_{h+1})) = \xi(\mathcal{S}(Tx_h, Tx_h, Tx_{h+1}), M_T(x_h, x_h, x_{h+1}))$$
$$\geq 0 \tag{2.2}$$

where

$$M_T(x_h, x_h, x_{h+1}) = max\{\ S(x_h, x_h, x_{h+1}), S(x_h, x_h, Tx_h), S(x_{h+1}, x_{h+1}, Tx_{h+1}),$$
$$\frac{S(x_h, x_h, Tx_{h+1}) + S(x_{h+1}, x_{h+1}, Tx_h)}{2}\}$$
$$= max\{S(x_h, x_h, x_{h+1}), S(x_h, x_h, x_{h+1}), S(x_{h+1}, x_{h+1}, x_{h+2}),$$
$$\frac{S(x_h, x_h, x_{h+2}) + S(x_{h+1}, x_{h+1}, x_{h+1})}{2}\}$$
$$\leq max\{S(x_h, x_h, x_{h+1}),\ S(x_{h+1}, x_{h+1}, Tx_{h+2}),$$
$$\frac{S(x_h, x_h, x_{h+1}) + S(x_{h+1}, x_{h+1}, x_{h+2})}{2}\}$$
$$= max\{S(x_h, x_h, x_{h+1}),\ S(x_{h+1}, x_{h+1}, Tx_{h+2})\ \}.$$

Hence

$$M_T(x_h, x_h, x_{h+1}) = max\{S(x_h, x_h, x_{h+1}),\ S(x_{h+1}, x_{h+1}, Tx_{h+2})\}.$$

Suppose that $(x_h, x_h, x_{h+1}) \leq S(x_{h+1}, x_{h+1}, Tx_{h+2})$ for some $h \in \mathbb{N}$.

Then, we have

$$M_T(x_h, x_h, x_{h+1})\ = max\{S(x_h, x_h, x_{h+1}),\ S(x_{h+1}, x_{h+1}, Tx_{h+2})\} = S(x_{h+1}, x_{h+1}, Tx_{h+2}).$$

Hence from (2.2), we have

$$0 \leq \xi(\ S(Tx_{h+1}, Tx_{h+1}, Tx_{h+2}), M_T(x_h, x_h, x_{h+1}))$$
$$= \xi(S(Tx_{h+1}, Tx_{h+1}, Tx_{h+2}), S(Tx_{h+1}, Tx_{h+1}, Tx_{h+2}))$$
$$< S(x_{h+1}, x_{h+1}, x_{h+2}) - S(Tx_{h+1}, Tx_{h+1}, Tx_{h+2})\ \leq 0,$$

a contradiction.

Hence $S(x_{h+1}, x_{h+1}, x_{h+2}) < S(x_h, x_h, x_{h+1})$ for all $h \in \mathbb{N}$.

Therefore, $\{S(x_h, x_h, Tx_{h+1})\ \}$ is decreasing and bounded below. So there exists $r \geq 0$ such that $\lim_{n\to\infty} S(x_h, x_h, x_{h+1}) = r$.

Suppose that $r > 0$.

Now, using condition (ξ_3), with $t_h = S(x_{h+1}, x_{h+1}, x_{h+2})$ and

$s_h = S(x_h, x_h, x_{h+1})$, we have

$0 \leq \lim_{h\to\infty} sup\ \xi(\ S(x_{h+1}, x_{h+1}, x_{h+2}), S(x_h, x_h, x_{h+1}) < 0,$

a contradiction.

Therefore, $r = 0$.

That is

$$\lim_{h\to\infty} S(x_h, x_h, x_{h+1}) = 0. \tag{2.3}$$

Now, we show that $\{x_h\}$ is a Cauchy sequence. Suppose that $\{x_h\}$ is not a Cauchy sequence.

Then there exists $\epsilon > 0$ and sequence of positive integers $\{h_k\}$ and $\{g_k\}$ such that $h_k > g_k \geq k$ satisfying

$$S(x_{g_k}, x_{g_k}, x_{h_k}) \geq \epsilon. \tag{2.4}$$

Let us choose the smallest h_k satisfying (2.4), then we have

$h_k > g_k \geq k$ with
$\mathcal{S}(x_{g_k}, x_{g_k}, x_{h_k}) \geq \epsilon$ and
$\mathcal{S}(x_{g_k}, x_{g_k}, x_{h_{k-1}}) < \epsilon$,
satisfying (i)-(iv) of Lemma 1.3.

Hence, we have

$$M_s(x_{g_k}, x_{g_k}, x_{h_k}) = max \left\{ \begin{array}{c} \mathcal{S}(x_{g_k}, x_{g_k}, x_{h_k}), \mathcal{S}(x_{g_k}, x_{g_k}, Tx_{g_k}), \mathcal{S}(x_{h_k}, x_{h_k}, Tx_{h_k}), \\ \frac{\mathcal{S}(x_{g_k}, x_{g_k}, Tx_{h_k}) + \mathcal{S}(x_{h_k}, x_{h_k}, Tx_{g_k})}{2} \end{array} \right\}$$

On taking limit as $k \to \infty$,

we have $M_s(x_{g_k}, x_{g_k}, x_{h_k}) = \epsilon$

Using condition (ξ_3) with

$t_k = \mathcal{S}(x_{g_{k+1}}, x_{g_{k+1}}, x_{h_{k+1}})$ and $s_k = M(x_{g_k}, x_{g_k}, x_{h_k})$,
we have $0 \leq \lim_{k \to \infty} sup\ \xi(\mathcal{S}(x_{g_{k+1}}, x_{g_{k+1}}, x_{h_{k+1}}), M_s(x_{g_k}, x_{g_k}, x_{h_k})) < 0$,

a contradiction.

Thus $\{x_h\}$ is a Cauchy sequence.

Since X is a S-metric space then, there exists $b \in X$ such that $\lim_{h \to \infty} x_h = b$. Since T is a β-γ-continuous and $\beta(x_h, x_h, x_{h+1}) \geq \gamma(x_h, x_h, x_{h+1})$ for all $h \in \mathbb{N}$, we have $b = \lim_{n \to \infty} x_{h+1} = \lim_{h \to \infty} Tx_h = T \lim_{h \to \infty} x_h = Tb$. Hence T has a fixed point.

In the following theorem, we replace the β-γ-continuity of T by another condition.

Theorem 2.5. Let $(X, \mathcal{S})$ be a S-metric space and $T : X \to X$ and $\beta, \gamma : X \times X \times X \to [0, \infty)$ be mappings.

Suppose that the following conditions are satisfied:

  I. T is a generalized β-γ-Z-contraction with respect to ξ,
  II. T is a β-admissible mapping with respect to γ,
  III. there exists $x_0 \in X$ such that $\beta(x_0, x_0, Tx_0) \geq \gamma(x_0, x_0, Tx_0)$,
  IV.   if $\{x_h\}$ is a sequence in X such that $\beta(x_h, x_h, x_{h+1}) \geq \gamma(x_h, x_h, x_{h+1})$ for all $h \in \mathbb{N}$ and $x_h \to x_1 \in X$ as $h \to \infty$, then there exists a subsequence $\{x_{h_p}\}$ of $\{x_h\}$ such that

$\beta(x_{h_p}, x_{h_p}, b) \geq \gamma(x_{h_p}, x_{h_p}, b)$ for all $p \in \mathbb{N}$.

Then $\{T^n x_0\}$ converges to an element b of X and b is a fixed point of T.

Proof. By using similar arguments as in the proof of Theorem 2.1, we obtain that the sequence $\{x_h\}$ defined by $x_{h+1} = Tx_h$ converges to $b \in X$ and $\beta(x_h, x_h, x_{h+1}) \geq \gamma(x_h, x_h, x_{h+1})$ for all $p \in \mathbb{N}$.

By (IV), there exists a subsequence $\{x_{h_p}\}$ of $\{x_h\}$ such that $\beta(x_{h_p}, x_{h_p}, b) \geq \gamma(x_{h_p}, x_{h_p}, b)$ for all $p \in \mathbb{N}$. Hence from (2.1) we have

$$0 \leq \xi(\mathcal{S}(Tx_{h_p}, Tx_{h_p}, Tb), M_T(x_{h_p}, x_{h_p}, b))$$
$$= \xi(\mathcal{S}(Tx_{h_p+1}, Tx_{h_{p+1}}, Tb), M_T(x_{h_p}, x_{h_p}, b))$$

$$< M_T(x_{h_p}, x_{h_p}, b) - S(Tx_{h_p+1}, Tx_{h_p+1}, Tb), \tag{2.5}$$

Which implies that

$$S(Tx_{h_p+1}, Tx_{h_p+1}, Tb) < M_T(x_{h_p}, x_{h_p}, b).$$

Now, we have

$$S(Tx_{h_p+1}, Tx_{h_p+1}, Tb) \leq S(Tx_{h_p+1}, Tx_{h_p+1}, Tb) < M_T(x_{h_p}, x_{h_p}, b) \tag{2.6}$$

and

$$
\begin{aligned}
S(b, b, Tb) &\leq M_T(x_{h_p}, x_{h_p}, b) \\
&= max \left\{ \begin{array}{l} S\left(x_{h_p}, x_{h_p}, b\right), S\left(x_{h_p}, x_{h_p}, Tx_{h_p}\right), S(b, b, Tb), \\ \frac{S\left(x_{h_p}, x_{h_p}, Tb\right) + S(Tx_{h_p}, Tx_{h_p}, b)}{2} \} \end{array} \right\} \\
&\leq max \left\{ \begin{array}{l} S\left(x_{h_p}, x_{h_p}, b\right), S\left(x_{h_p}, x_{h_p}, Tx_{h_p}\right), S(b, b, Tb), \\ \frac{S\left(x_{h_p}, x_{h_p}, x^1\right) + S(b,b,Tb) + S(Tx_{h_p}, Tx_{h_p}, b)}{2} \} \end{array} \right\}
\end{aligned}
$$

On taking limits as $h \to \infty$ we have

$$S(b, b, Tb) \leq \lim_{p \to \infty} M_T(x_{h_p}, x_{h_p}, b) \leq S(b, b, Tb).$$

Therefore

$$\lim_{p \to \infty} M_T(x_{h_p}, x_{h_p}, b) = S(b, b, Tb).$$

From (2.6), we have

$$
\begin{aligned}
c &\leq S(b, b, Tx_{h_p}) + S(x_{h_p}, x_{h_p}, Tb) \\
&\leq S(b, b, Tx_{h_p}) + M_T(x_{h_p}, x_{h_p}, b), \tag{2.7}
\end{aligned}
$$

on taking limit as $p \to \infty$ on (2.7), we have

$$
\begin{aligned}
S(b, b, Tb) &\leq \lim_{p \to \infty} S(x_{h_p+1}, x_{h_p+1}, Tb) \\
&\leq S(b, b, Tb). \tag{2.8}
\end{aligned}
$$

Hence, we have

$$\lim_{p \to \infty} S(x_{h_p+1}, x_{h_p+1}, Tb) = S(b, b, Tb). \tag{2.9}$$

Suppose $b \neq Tb$.

Now we choose $c_p = S(x_{h_p+1}, x_{h_p+1}, Tb)$ and $d_p = M_T(x_{h_p}, x_{h_p}, b)$.

From property (ξ_3), it follows that

$$0 \leq \lim_{p \to \infty} sup \xi (S(Tx_{h_p}, Tx_{h_p}, Tb), M_T(x_{h_p}, x_{h_p}, b) < 0,$$

a contradiction.

Hence $Tb = b$.

Therefore T has a fixed point.

Theorem 2.6. In addition to the hypotheses of Theorem 2.2(Theorem 2.3), assume the following condition (A): for all $x \neq y \in X$, there exists $w \in X$ such that $\beta(x, x, w) \geq \gamma(x, x, w)$, $\beta(y, y, w) \geq \gamma(y, y, w)$ and $\beta(w, w, Tw) \geq \gamma(w, w, Tw)$. Then T has a unique fixed point.

Proof. Suppose that c and b are two fixed points of T with $c \neq b$. Then by our assumption, there exists $w \in X$ such that $\beta(c, c, w) \geq \gamma(c, c, w)$, $\beta(b, b, w) \geq \gamma(b, b, w)$ and $\beta(w, w, Tw) \geq \gamma(w, w, Tw)$ so that condition (III) of Theorem 2.2(Theorem 2.3) holds with $x_0 = w$, also. Now, by applying Theorem 2.2(Theorem 2.3), we deduce that $\{T^h w\}$ converges to a fixed point a of T and hence the sequence is $\{S(a, a, T^h w)\}$ is bounded.

Now, since $S(c, c, T^h w) \leq [S(c, c, a) + S(a, a, T^h w)]$, we have the sequence $S(c, c, T^h w)$ is bounded. Therefore there exists a sub-sequence $\{S(c, c, T^{h_p} w)\}$ of $S(c, c, T^h w)$ such that $\lim_{h \to \infty} S(c, c, T^{h_p} w) = r$, for some non-negative real r.

Now, we have

$$
\begin{aligned}
S(c, c, T^{h_p} w) &\leq M_T(c, c, T^{h_p} w) \\
&= \max \left\{ \begin{array}{c} S(c, c, T^{h_p} w), S(T^{h_p} w, T^{h_p} w, T^{h_{p+1}} w), \\ \frac{S(z^1, z^1, T^{h_{p+1}} w) + S(Tz^1, Tz^1, T^{h_p} w)}{2} \end{array} \right\} \\
&= \max \left\{ \begin{array}{c} S(c, c, T^{h_p} w), S(T^{h_p} w, T^{h_p} w, T^{h_{p+1}} w), \\ \frac{S(c, c, T^{h_{p+1}} w) + S(c, c, T^{h_p} w)}{2} \end{array} \right\} \qquad (2.10) \\
&\leq \max \left\{ \begin{array}{c} S(c, c, T^{h_p} w), S(T^{h_p} w, T^{h_p} w, T^{h_{p+1}} w), \\ \frac{[S(c, c, T^{h_p}) + S(T^{h_p} w, T^{h_p} w, T^{h_{p+1}} w)] + S(c, c, T^{h_p} w)}{2} \end{array} \right\}
\end{aligned}
$$

On taking limits as $p \to \infty$ we have $\lim_{h \to \infty} M_T(c, c, T^{h_p} w) = r$.

Now we show that $r = 0$. Suppose $r > 0$.

Since T is β-admissible with respect to γ, we have

$\beta(w, w, T^h w) \geq \gamma(w, w, T^h w)$

and hence $\beta(c, c, T^h w) \geq \gamma(c, c, T^h w)$

$\beta(c, c, T^h w) \geq \gamma(b, b, T^h w)$ for all $h \in \mathbb{N}$.

Now from (2.1), we have

$\xi\{S(c, c, T^{h_{p+1}} w), M_T(c, c, T^{h_p} w)\} \geq 0$.

Hence, we have $S(c, c, T^{h_{p+1}} w) \leq M_T(c, c, T^{h_p} w)$

which implies that

$$
\begin{aligned}
S(c, c, T^{h_{p+1}} w) &\leq S(c, c, T^{h_{p+1}} w) \\
&\leq M_T(c, c, T^{h_p} w).
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
S(c, c, T^{h_p} w) &\leq S(c, c, T^{h_{p+1}} w) + S(c, c, T^{h_{p+1}} w) + S(T^{h_p} w, T^{h_p} w, T^{h_{p+1}} w) \\
&= 2 S(c, c, T^{h_{p+1}} w) + S(T^{h_p} w, T^{h_p} w, T^{h_{p+1}} w)
\end{aligned}
$$

$$\leq 2\,S(c,c,T^{h_{p+1}}w)\; +\; S(\,T^{h_{p+1}}w,T^{h_{p}+1}w,T^{h_{p}}w)$$
$$\leq M_T(c,c,w)\; +\; S(c,c,T^{h_{p}}w).$$

On taking limits as $p \to \infty$ we have

$$\lim_{h\to\infty}\; S(c,c,T^{h_{p+1}}w) = r.$$

Now, by choosing $t_p\; =\; S(c,c,T^{h_{p+1}}w)$ and $s_p\; =\; M_T(c,c,T^{h_{p}}w),$

from property (ξ_3), it follows that

$$0 \leq \lim_{p\to\infty} sup\xi(\,S(c,c,T^{h_{p+1}}w),\, M_T(c,c,T^{h_{p}}w)\,) < 0,$$

a contradiction.

Hence $r = 0$. Hence $T^{h_{p}}w \to c$ as $h \to \infty$.

Therefore $c = a$.

Similarly we can prove that $b = a$.

Thus it follows that $c = b$,

a contradiction.

Hence T has a unique fixed point.

Example 2.7. Let $X = [0,2]$ and $S : [0,2] \times [0,2] \times [0,2] \to [0,\infty)$ as

$$S(x,y,z) = |x - z| + |x + z - 2y|\,.$$

Then, clearly (X,S) is a complete metric space.

Now, define $Tx = \begin{cases} \frac{x}{6}\,, x \in [0,1) \\ 2x - 2,\; x \in [1,2] \end{cases},$

$$\beta(x,y,z) = \begin{cases} 2 + xyz\,,\; x,y,z \in [0,1] \\ 0 \qquad\qquad otherwise \end{cases}$$

and

$$\gamma(x,y,z) = \begin{cases} 1 + xyz\,,\; x,y,z \in [0,1] \\ 4 \qquad\qquad otherwise \end{cases}$$

Taking, $\xi(t,s) = \frac{s}{1+s} - t.$

Clearly, T is β-admissible mapping with respect to γ and β-γ continuous mapping.

For all $x,y,z \in [0,1)$

$$\beta(x,y,z) \geq \gamma(x,y,z).$$

$$\xi(S(Tx,Tx,Ty),\, M_T(x,x,y)) = \frac{M_T(x,x,y)}{1+ M_T(x,x,y)} - S(Tx,Tx,Ty)$$
$$\geq \frac{S(x,x,y)}{1+ S(x,x,y)} - S(Tx,Tx,Ty)$$

$$= \frac{2|x-y|}{1+2|x-y|} - \frac{1}{3}|x-y|$$
$$\geq 0.$$

So, all the conditions of Theorem 2.4 are hold.

Hence T has fixed points. Clearly, 0 and 1 are the fixed points.

If in the above setting we redefine $\beta(x,y,z) = 2$ and $\gamma(x,y,z) = 1$ for all $x,y,z \in X$.

If $Tx = \frac{x}{7}$ for all $x \in [0,2]$, then all the conditions of Theorem 2.6 are satisfied, so T has a unique fixed point. Clearly, 0 is the unique fixed point of T.

3 A Fixed Point Result in Partially Ordered S-metric Spaces

Definition 3.1. Let $(X, \succcurlyeq)$ be a partially ordered set. If there exists a S-metric S on X such that (X, S) is complete, then we say that $(X, \succcurlyeq, S)$ is a partially ordered complete S-metric space.

Theorem 3.2. Let $(X, \succcurlyeq, S)$ be a partially ordered complete S-metric space and $T : X \to X$ be a self-map of X. Assume that the following conditions are satisfied:

(i)   there exists a simulation mapping ξ such that

$\xi(S(Tx, Tx, Ty), M_T(x,y)) \geq 0$, for any $x, y \in X$ with $x \succcurlyeq y$,

Where $M_T(x, x, y) = max\{ S(x,x,y), S(x,x,Tx), S(y,y,Ty), \frac{S(x,x,Ty)+S(y,y,Tx)}{2} \}$,

(ii) T is a non-decreasing.
(iii)   There exists an $x_0 \in X$ such that $x_0 \succcurlyeq Tx_0$.
(iv)   Either T is continuous or $\{x_h\}$ is a decreasing sequence with $x_h \to a$ as $h \to \infty$, then there exists a subsequence $\{x_{h_p}\}$ of $\{x_h\}$ such that $x_{h_p} \succcurlyeq x_0$ for all $p \in \mathbb{N}$.

Then $\{T^h x_0\}$ converges to an element a of X is a fixed point of T.

Further, if for all $x \neq y \in X$, there exists $w \in X$ such that $x \succcurlyeq w$, $y \succcurlyeq w$ and $w \succcurlyeq Tw$, then T has a unique fixed point.

Proof. We define functions $\beta, \gamma : X \times X \times X \to [0, \infty)$ by
$$\beta(x,y) = \{ \begin{smallmatrix} 3 \ if \ x \succcurlyeq y \\ 0 \ otherwise \end{smallmatrix}$$
and

$$\gamma(x,y) = \{ \begin{smallmatrix} 1 \ if \ \ x \succcurlyeq y \\ 4 \ otherwise \end{smallmatrix} \ .$$

Now, for any $x, y \in X$, $\beta(x,x,y) \geq \gamma(x,x,y)$ if and only if $x \succcurlyeq y$.

By (i), we have

$\xi(S(Tx, Tx, Ty), M_T(x,x,y)) \geq 0$.

Suppose that $\beta(x, x, Tx) \geq \gamma(x, x, Tx)$,

then we have

$x \succcurlyeq Tx$. Since T is non-decreasing, we have $Tx \succcurlyeq TTx$

which implies that

$\beta(Tx, TTx) \geq \gamma(Tx, TTx)$, hence T is β-admissible with respect to γ.

Further, suppose that $\beta(x, x, y) \geq \gamma(x, x, y)$ and $\beta(y,, y, Ty) \geq \gamma(y, y, Ty)$, so that we have $x \succcurlyeq y$ and $y \succcurlyeq Ty$. It follows that $x \succcurlyeq Ty$ and hence $\beta(x, x, Ty) \geq \gamma(x, x, Ty)$. Thus T is β-admissible with respect to γ. Hence T satisfy all the hypotheses of Theorem 2.1 (Theorem 2.2) and T has a fixed point.

Moreover, if for all $x \neq y \in X$, there exists $w \in X$, such that $x \succcurlyeq w$, $y \succcurlyeq w$ and $w \succcurlyeq Tw$, then we have $\beta(x, x, w) \geq \gamma(x, x, w)$, $\beta(y, y, w) \geq \gamma(y, y, w)$ and $\beta(w, w, Tw) \geq \gamma(w, w, Tw)$.

Hence by Theorem 2.3, T has a unique fixed point.

4 Corollaries

In this section, we shall derive some results of literature from our main results.

Corollary 4.1. Let $(X, \mathcal{S})$ be a S-metric space. Let $T : X \to X$ and $\beta, \gamma : X \times X \times X \to [0, \infty)$ be mappings. Suppose that the following conditions are satisfied:

(i) There exists a simulation mapping ξ such that for any $x, y \in X$, $\beta(x, x, y) \geq \gamma(x, x, y)$ implies $\xi(\mathcal{S}(Tx, Tx, Ty), \mathcal{S}(x, x, y)) \geq 0$.
(ii) is a β-admissible mapping with respect to γ.
(iii) There exists an $x_1 \in X$ such that $\beta(x_1, x_1, Tx_1) \geq \gamma(x_1, x_1, Tx_1)$, and
(iv) T is an β-γ-continuous mapping or if $\{x_h\}$ is a sequence in X such that $\beta(x_h, x_h, x_{h+1})$

$\gamma(x_h, x_h, x_{h+1})$ for all $h \in \mathbb{N}$ and $x_h \to a \in X$ as $h \to \infty$, then there exists a sub-sequence $\{x_{h_p}\}$ of $\{x_h\}$ such that $\beta(x_{h_p}, x_{h_p}, a) \geq \gamma(x_{h_p}, x_{h_p}, a)$ for all $p \in \mathbb{N}$.

Then T has a fixed point $a \in X$ and $\{T^h x_1\}$ converges to a.

Moreover, if for all $x \neq y \in X$, there exists $w \in X$ such that $\beta(x, x, w) \geq \gamma(x, x, w)$, $\beta(y, y, w) \geq \gamma(y, y, w)$ and $\beta(w, w, Tw) \geq \gamma(w, w, Tw)$, then T has a unique fixed point.

Proof. By taking $M(x, x, y) = \mathcal{S}(x, x, y)$ proof follows from Theorem 2.5.

Corollary 4.2. Let $(X, \mathcal{S})$ be a S-metric space. Let $T : X \to X$ and $\beta, \gamma : X \times X \times X \to [0, \infty)$ be mappings. Assume that there exists two continuous function $\mu, \rho : [0, \infty) \to [0, \infty)$ with $\mu(t) < t \leq \rho(t)$ for all $t > 0$ and $\mu(t) = \rho(t) = 0$ if and only if $t = 0$ such that for any $x, y \in X$ with $\beta(x, x, y) \geq \gamma(x, x, y)$ implies

$$\rho(\mathcal{S}(Tx, Tx, Ty) \leq \mu(M_T(x, x, y)) \tag{4.1}$$

Where $M_T(x, x, y) = max\{\mathcal{S}(x, x, y), \mathcal{S}(x, x, Tx), \mathcal{S}(y, y, Ty), \frac{\mathcal{S}(x, x, Ty) + \mathcal{S}(y, y, Tx)}{2}\}$.

Suppose that the following conditions are satisfied:

(i) T is a β-admissible mapping;
(ii) There exists $x_1 \in X$ such that $\beta(x_1, x_1, Tx_1) \geq \gamma(x_1, x_1, Tx_1)$ and
(iii) Either T is an β-γ-continuous mapping, or if $\{x_h\}$ is a sequence in X such that

$\beta(x_h, x_h, x_{h+1}) \geq \gamma(x_h, x_h, x_{h+1})$ for all $h \in \mathbb{N}$ and $x_h \to a \in X$ as $h \to \infty$, then there exists a subsequence $\{x_{h_p}\}$ of $\{x_h\}$ such that $\beta(x_{h_p}, x_{h_p}, a) \geq \gamma(x_{h_p}, x_{h_p}, a)$ for all $p \in \mathbb{N}$.

Then $\{T^n x_1\}$ converges to an element x_1 of X and x_1 is a fixed point of T.

Proof. The proof of this corollary follows from Theorem 2.4(Theorem 2.5) by taking $\xi(t,s) = \mu(s) = \rho(t)$ for all $t, s \geq 0$.

Corollary 4.3. Let $(X, \mathcal{S})$ be a S-metric space with $T : X \to X$ and $\beta, \gamma : X \times X \times X \to [0, \infty)$ be mappings. Suppose that the following conditions are satisfied:

(i)   There exists a simulation mapping ξ such that for any $x, y \in X$ with $\beta(x, x, y) \geq 1$, implies $\xi(\mathcal{S}(Tx, Tx, Ty), (M_T(x, x, y)) \geq 0$, where

$$M_T(x, x, y) = max\{\ \mathcal{S}(x, x, y), \mathcal{S}(x, x, Tx), \mathcal{S}(y, y, Ty), \frac{\mathcal{S}(x,x,Ty)+\mathcal{S}(y,y,Tx)}{2}\ \}.$$

(ii) T is a β-admissible mapping,

(iii)   There exists an $x_1 \in X$ such that $\beta(x_1, x_1, Tx_1) \geq 1$, and

(iv) T is an β-continuous mapping, or if $\{x_h\}$ is a sequence in X such that $\beta(x_h, x_h, Tx_{h+1}) \geq 1$ for all $h \in X$, and $x_h \to a \in X$ as $h \to \infty$, then there exists subsequence $\{\ x_{h_p}\ \}$ of $\{x_h\}$ such that $\beta(x_{h_p}, x_{h_p}, a) \geq 1$ for all $p \in \mathbb{N}$.

Then T has a fixed point $a \in X$ and $\{T^h x_1\}$ converges to a.

Moreover, if for all $x \neq y \in X$, there exists $w \in X$ such that $\beta(x, x, w) \geq 1$, then T has a unique fixed point.

Proof. Follows from Theorem 2.4(Theorem 2.5) and Theorem 2.6 by taking $\gamma(x, x, y) = 1$ for all $y \in X$.

5 Conclusion

In this manuscript, we have introduced the new notion of generalized β-γ-Z contraction mapping with respect to a simulation function ξ and proved fixed point theorems for such contractions in complete S-metric spaces. Further, we have extended it to partially ordered complete S-metric spaces. In the end, we have provided some corollaries of our main results.

Competing Interests

Authors have declared that no competing interests exist.

References

[1]   Banach S. Sur les operations dans les ensembles abstraits et leur application aux equations integrals. Fund. Math. 1922;3:133-181.

[2]   Babu GVR, Dula TM. Fixed points of almost generalized $(\alpha, \beta) - (\psi, \varphi)$-contractive mappings in b-metric spaces. (Communicated).

[3]   Chuadchawna P, Kaewcharoen A, Plubtieng S. Fixed point theorems for generalized α-η-ψ-Gerghty contraction type mappings in $\alpha - \eta$-complete metric spaces. J. Nonlinear Sci. Appl. 2016;9:471-485.

[4]   Hussain N, Kutbi MA, Salimi P. Fixed Point theory in α-complete metric spaces with applications. Abstr. Appl. Anal. 2014;2014:11.

[5]   Karapinar E, Kumam P, Salimi P. On α-ψ Meir-Keeler contractive mappings. Fixed Point Theory Appl. 2013;2013:12 .

[6]   Samet B, Vetro C, Vetro P. Fixed point theorem for α-ψ-contractive type mappings. Nonlinear Anal. 2012;75:2154-2165.

[7]   Ansari AH, Chandok S, Ionescu C. Fixed point theorems on metric spaces for weak contractions with auxiliary functions. Journal of Inequalities and Applications. 2014;2014: 429.

[8]   Argoubi H, Samet B, Vetro C. Nonlinear contractions involving simulation functions in a metric space with a partial order. J. Nonlinear Sci. Appl. 2015;8:1082-1094.

[9]   Babu GVR, Sailaja PD. A Fixed Point Theorem of Generalized Weakly Contractive Maps in Orbitally Complete Metric Spaces. Thai Journal of Mathematics. 2011;9(1):1-10.

[10]   Berinde V. Approximating fixed points of weak contractions using the Picard iteration, Nonlinear. Anal. Forum. 2004;9:43-53.

[11]   Geraghty M. On contractive mappings. Proc. Amer. Math. Soc. 1973;40:604-608.

[12]   Hussain N. Salimi P. Suzuki-Wardowski type fixed point theorems for α-GF-contractions. J. Taiwanese. 2014;18(6):1879-1895.

[13]   Khojasteh F, Shukla S, Radenovic S. A new approach to the study of fixed point theorems via simulation functions. Filomat. 2015;29:1189-1194.

[14]   Olgun M, Bicer O, Alyildiz T. A new aspect to Picard operators with simulation functions. J. Turk. Math. 2016;40:832-837.

[15]   Sedghi S, Shobe N, Aliouche A. A generalization of fixed point theorems in S-metric spaces. Mat. Vesnik. 2012;64:258-266.

[16]   Sedghi S, Dung NV. Fixed Point Theorems on S-Metric Spaces. Mat. Vesnik. 2014;66(1):113-124.

[17]   Popescu O. Some new fixed point theorems for α-Geraghty contraction type maps in metric spaces. Fixed Point Theory Appl. 2014;2014:12.

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