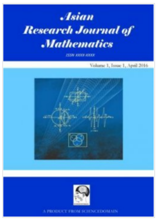

Asian Research Journal of Mathematics

8(2): 1-11, 2018; Article no.ARJOM.38351

ISSN: 2456-477X

Extension of Moore-Penrose Pseudoinverse to Solve Nonsquare Fuzzy System of Linear Equations

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/ARJOM/2018/38351

Editor(s):

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Complete Peer review History: http://www.sciencedomain.org/review-history/22746

Received: ***23[rd] October 2017***

Accepted: ***6[th] January 2018***

Published: ***15[th] January 2018***

Original Research Article

Abstract

In this work, a solution for the fuzzy system of linear equations has been assessed. Here, the coefficients matrix is nonsquare with elements of crisp numbers such that the known and unknown vectors are fuzzy vectors. First, the singular value decomposition and the generalized inverse of the nonsquare matrices have been illustrated. Then, these concepts have been extended to find the solutions of the nonsquare fuzzy system of linear equations.

Keywords: Fuzzy system of linear equations; Fuzzy numbers; Moore-Penrose pseudoinverse; singular value decomposition.

2010 Mathematics Subject Classification: 15A06, 15A09, 15A18.

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1 Introduction

A system of linear equations is a set of linear equations in several variables which is a basic part of linear algebra. The importance of this theory is that we often deal with a large proportion of problems in many branches of science such as physics, engineering, economic and etc. In such cases, linear algebra or computational methods in numerical linear algebra occurs as a strong tool. The system of linear equations can be represented in a matrix form [1]. In many problems, this matrix appears in a nonsquare form, where in this situations, Moore-Penrose pseudoinverse will be effective. This matrix and its inverse was first commended by Moore in 1920 and Penrose in 1955, independently and known as the generalized inverse, Moore-Penrose inverse or Moore-Penrose pseudoinverse. Moreover, in some applications, can be seen that the parameters are performed by fuzzy number rather than crisp. Accordingly, fuzzy system of linear equations is an interesting and practical topic in linear algebra which has developed rapidly [2, 3]. Recent years have seen a development in use of fuzzy mathematics in science and technology requiring the increasing importance of study and development on mathematical models and numerical methods for fuzzy system of linear equations [4, 5]. Over the last few years, many authors have been interested in working on this topic. It is worth mentioning that Zadeh [6], Dubois and Prade [7] were the pioneers of fuzzy numbers and fuzzy arithmetic operations. Many other authors have made much effort to extend the concept of fuzzy numbers to other branches of science, in particular the fuzzy system of linear equations. For instance, Friedman, Ming and Kandel have considered an $n \times n$ fuzzy system of linear equations and turned this system into a $2n \times 2n$ crisp system of linear equations [8]. Allahviranloo has applied multifarious approaches to solve the fuzzy system of linear equations [9, 10]. Vroman and his colleagues have used parametric functions to solve the fuzzy system of linear equations with a symmetric matrix [11]. The fully fuzzy system of linear equations has been survived by Allahviranloo and his colleagues [12]. Ezzati has developed an approach to solve an arbitrary $m \times n$ daul fuzzy system of linear equations as $A\widetilde{x} = B\widetilde{x} + \widetilde{y}$ which $m \leq n$ and $\widetilde{y}$ is a symmetric fuzzy number vector [13].

In this article, a general method is devised for solving an $m \times n$ fuzzy system of linear equations whose coefficients matrix is crisp and the right-hand side column is an arbitrary fuzzy number vector. In section 2, singular value decomposition and generalized inverse are introduced. In section 3, first the basic concepts of fuzzy mathematics is presented and then the solution of the $m \times n$ fuzzy system of linear equations is explicated. Finally, in section 4, some numerical examples is served to illustrate our method.

2 Singular Value Decomposition and Moore-Penrose Pse-udoinverse

Let $\mathrm{M}_{m\times n}$ denote the set of all $m \times n$ matrices over the field of complex numbers. The symbols A^T and A^* stand for transpose and conjugate transpose (or Hermitian transpose), respectively. In this section, the system of linear equations $Ax = y$ is studied, where A is a given $m \times n$ matrix, x and y are column vectors with n and m components, respectively. In some problems, there exists no solution or no unique solution for this system. In this situation, we find a vector x such that $\|Ax-y\|$ is the smallest possible in terms of least squares. To study this topic, we deal with concept of generalized inverse. Let us first recall some useful notations of linear algebra and then introduce the concept of generalized inverse or Moore-Penrose pseudoinverse.

Definition 2.1. [14] A symmetric matrix A is positive definite if for each nonzero vector x, $x^T Ax > 0$.

Definition 2.2. [14] Let $A \in \mathrm{M}_{m\times n}$ and $B = A^*A$. The singular values of A are $\sigma_i = \sqrt{\lambda_i}$, $i = 1, 2, \ldots, n$, where λ_i are eigenvalues of B.

Note that B is an $n \times n$ symmetric positive definite matrix. Therefore, its eigenvalues are nonnegative.

Definition 2.3. [15] An orthogonal matrix is a real square matrix whose columns and rows are orthogonal vectors.

Definition 2.4. [15] Let A be a real square matrix. Then, A is said to be an orthonormal matrix if $A^T A = I$, where I is the identity matrix.

Definition 2.5. [15] Let A be a complex square matrix. Then, A is said to be an unitary matrix if $A^* A = I$, where I is the identity matrix.

Theorem 2.1. *[14] Let $A \in M_{m\times n}$ be any matrix, and $\sigma_1 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = \sigma_n = 0$ are nonzero singular values of A with $r = \min(m,n)$. Then, there exist unitary matrices $U = [u_1, \ldots, u_m] \in M_{m\times m}$ and $V = [v_1, \ldots, v_n] \in M_{n\times n}$ such that $U^* AV = \Sigma$, where Σ is given by*

$$\Sigma = diag(\sigma_1, \ldots, \sigma_r, \ldots, 0)_{m\times n}. \tag{2.1}$$

Remark 2.1. Note that the factorization of the form

$$A = U\Sigma V^* \tag{2.2}$$

is called the singular value decomposition of A abbreviated by SVD. The m columns of U and n columns of V are respectively the eigenvectors of AA^* and A^*A and called the left and right singular vectors of A. The singular values of A, that is, σ_i are uniquely determined and the vectors u_i and v_i, the columns of the matrices U and V in (2.2), are associated with σ_i. This fact leads us to the existence and uniqueness of SVD stated in the following theorem.

Theorem 2.2. *[14] Each matrix A has a SVD of the form* (2.2)*. The singular values σ_i are uniquely determined, and if A is square and σ_i are distinct, then u_i and v_i, the columns of the matrices U and V in* (2.2)*, are uniquely determined up to complex signs.*

Definition 2.6. [16] Let $A \in \mathrm{M}_{m\times n}$, then there exists the unique matrix $A^+ \in \mathrm{M}_{n\times m}$ satisfying the following conditions:

1. $AA^+A = A$,
2. $A^+AA^+ = A^+$,
3. $A^+A = (A^+A)^*$,
4. $AA^+ = (AA^+)^*$.

The matrix A^+ is called the Moore-Penrose pseudoinverse, and the above four conditions are well known as Penrose conditions.

Remark 2.2. Note that the Moore-Penrose pseudoinverse exists and is unique for any matrix $A \in \mathrm{M}_{m\times n}$. A simple way to compute the Moore-Penrose pseudoinverse is by using SVD. Let $A \in \mathrm{M}_{m\times n}$ be any arbitrary matrix with rank r and SVD of the form (2.2), then the Moore-Penrose pseudoinverse of A is assessed by $A^+ = V\Sigma^+U^* = V\,\mathrm{diag}(\sigma_1^{-1}, \ldots, \sigma_r^{-1}, \ldots, 0)_{m\times n}U^*$. If $A \in M_{n\times n}$ is invertible, then $r = rank(A) = n$ and hence $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ and $\Sigma^+ = \mathrm{diag}(\sigma_1^{-1}, \ldots, \sigma_n^{-1})$ implying that $\Sigma^+ = \Sigma^{-1}$. Therefore, $A^+ = V\Sigma^+U^* = V\Sigma^{-1}U^* = (U\Sigma V^*)^{-1} = A^{-1}$.

Definition 2.7. [17] For any linear system $Ax = y$, the least-squares solution (l.s.s) of $Ax = y$ is finding a vector $\widehat{x}$ such that

$$\|A\widehat{x} - y\| = \min_x \|Ax - y\|, \tag{2.3}$$

where $\|\cdot\|$ is a norm in Euclidean space.

Theorem 2.3. *[14] Let $A \in M_{m\times n}$. The least-squares solution of the smallest norm of the linear system $Ax = b$ is given by*

$$\widehat{x} = A^+b = V\Sigma^+U^*b. \tag{2.4}$$

3 Solving the Nonsquare Fuzzy System of Linear Equations

In this section, we first review some main definitions and introduce the nonsquare fuzzy system of linear equations, and then we extend the Moore-Penrose pseudoinverse to solve our problem.

Definition 3.1. [18] Any arbitrary fuzzy number in the parametric form is represented by $(\underline{u}(r), \overline{u}(r))$, $0 \leq r \leq 1$ satisfying the following conditions.

1. $\underline{u}(r)$ is a bounded left-continuos non-decreasing function over [0,1],
2. $\overline{u}(r)$ is a bounded left-continuos non-increasing function over [0,1],
3. $\underline{u}(r) \leq \overline{u}(r), \quad 0 \leq r \leq 1$.

In particular, if $\underline{u},\ \overline{u}$ are linear functions, we have a triangular fuzzy number. A crisp number u is represented by $\underline{u}(r) = \overline{u}(r) = u,\ 0 \leq r \leq 1$. The set of all fuzzy numbers is denoted by E and any fuzzy number by the symbol of $'\sim'$.

Definition 3.2. [18] For any fuzzy numbers $\widetilde{u} = (\underline{u}(r), \overline{u}(r))$, $\widetilde{v} = (\underline{v}(r), \overline{v}(r))$, $0 \leq r \leq 1$ and real number k, the algebraic operations are defined as:

1. $k\widetilde{u} = \begin{cases} (k\underline{u}, k\overline{u}), & k \geq 0, \\ (k\overline{u}, k\underline{u}), & k < 0, \end{cases}$
2. $\widetilde{u} + \widetilde{v} = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r))$,
3. $\widetilde{u} - \widetilde{v} = (\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r))$.

In the reminder of this section, The non squares fuzzy system of linear equations are considered as follow:

$$\begin{cases} a_{11}(\underline{x}_1(r), \overline{x}_1(r)) + \cdots + a_{1n}(\underline{x}_n(r), \overline{x}_n(r)) = (\underline{y}_1(r), \overline{y}_1(r)) \\ \qquad\vdots \\ a_{m1}(\underline{x}_1(r), \overline{x}_1(r)) + \cdots + a_{mn}(\underline{x}_n(r), \overline{x}_n(r)) = (\underline{y}_m(r), \overline{y}_m(r)) \end{cases} \tag{3.1}$$

Where the crisp matrix $A = (a_{ij}) \in \mathrm{M}_{m \times n}$ is called coefficients matrix and the fuzzy number vector $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \cdots, \widetilde{x}_n)^T$ with components $\widetilde{x}_i = (\underline{x}_i(r), \overline{x}_i(r)),\ 1 \leq i \leq n,\ 0 \leq r \leq 1$ is a solution of fuzzy linear system (3.1) if for $i = 1, \ldots, m$

$$\underline{\sum_{j=1}^{n} a_{ij} x_j} = \sum_{j=1}^{n} \underline{a_{ij} x_j} = \underline{y}_i,$$
$$\overline{\sum_{j=1}^{n} a_{ij} x_j} = \sum_{j=1}^{n} \overline{a_{ij} x_j} = \overline{y}_i.$$

The authors Friedman et al. [8] converted the fuzzy linear system (3.1) to a $2m \times 2n$ system of linear equations as $S\widetilde{X} = \widetilde{Y}$, where

$$S = (s_{ij}), \quad 1 \leq i \leq 2m,\ 1 \leq j \leq 2n,$$
$$\widetilde{X} = (\underline{x}_1, \cdots, \underline{x}_n, \overline{x}_1, \cdots, \overline{x}_n)^T,$$
$$\widetilde{Y} = (\underline{y}_1, \cdots, \underline{y}_n, \overline{y}_1, \cdots, \overline{y}_n)^T.$$

and s_{ij} meet the following statements:

$$\begin{cases} a_{ij} \geq 0 \;\Rightarrow\; s_{ij} = s_{i+m,j+n} = a_{ij}, \\ a_{ij} < 0 \;\Rightarrow\; s_{i+m,j} = s_{i,j+n} = a_{ij}, \end{cases} \tag{3.2}$$

where any s_{ij} which is not determined by (3.2) will equal to zero. We can write

$$\begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix} \begin{pmatrix} \underline{x} \\ \overline{x} \end{pmatrix} = \begin{pmatrix} \underline{y} \\ \overline{y} \end{pmatrix}, \tag{3.3}$$

where

$$\underline{x} = (\underline{x}_1, \cdots, \underline{x}_n)^T, \quad \overline{x} = (\overline{x}_1, \cdots, \overline{x}_n)^T,$$
$$\underline{y} = (\underline{y}_1, \cdots, \underline{y}_n)^T, \quad \overline{y} = (\overline{y}_1, \cdots, \overline{y}_n)^T,$$

and

$$S_1 = \begin{pmatrix} s_{11} & \cdots & s_{1n} \\ s_{m1} & \cdots & s_{mn} \end{pmatrix}, \quad S_2 = \begin{pmatrix} s_{1,n+1} & \cdots & s_{1,2n} \\ s_{m,n+1} & \cdots & s_{m,2n} \end{pmatrix}.$$

Theorem 3.1. *[8] The matrix S is nonsingular if and only if the matrix $A = S_1+S_2$ and $B = S_1-S_2$ are both nonsingular.*

Theorem 3.2. *[8] The pseudo-inverse of non-negative matrix S is*

$$S^+ = \begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

where

$$A = \frac{1}{2}[(S_1+S_2)^+ + (S_1-S_2)^+] \ , \ B = \frac{1}{2}[(S_1+S_2)^+ - (S_1-S_2)^+].$$

Definition 3.3. [8] Let $\widetilde{X} = (\underline{x}_i(r), \overline{x}_i(r)),\ i = 1, \cdots, n$ denotes the unique solution of $S\widetilde{X} = \widetilde{Y}$. The fuzzy number vector $\widetilde{U} = (\underline{u}_i(r), \overline{u}_i(r)),\ i = 1, \cdots, n$ defined by

$$\underline{u}_i(r) = \min\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1)\}$$
$$\overline{u}_i(r) = \max\{\underline{x}_i(r), \overline{x}_i(r), \underline{x}_i(1)\}$$

is called the fuzzy solution of $S\widetilde{X} = \widetilde{Y}$. Furthermore, If $\underline{x}_i(r),\ \overline{x}_i(r),\ 1 \leq i \leq n$ are all fuzzy numbers, then

$$\underline{u}_i(r) = \underline{x}_i(r) \quad \text{and} \quad \overline{u}_i(r)) = \overline{x}_i(r), \quad 1 \leq i \leq n,$$

and $\widetilde{U}$ is called the strong fuzzy solution. Otherwise, $\widetilde{U}$ is the weak fuzzy solution.

By using Equation (2.4), the solution of system $S\widetilde{X} = \widetilde{Y}$ can be expressed by $\widetilde{X} = S^+\widetilde{Y}$. Where, S^+ is the Moore-Penrose pseudoinverse of S. Definition 3.3 is exerted to find weak and strong fuzzy solution.

4 Numerical Examples

In this section, we present some numerical examples to illustrate our method.

Example 4.1. *[19] Solve the following 3×2 fuzzy system of linear equations.*

$$\begin{cases} \widetilde{x}_1 + \widetilde{x}_2 = (r, 2-r) \\ \widetilde{x}_1 - \widetilde{x}_2 = (1+r, 3-r) \\ 2\widetilde{x}_1 + \widetilde{x}_2 = (2r, 3-r) \end{cases}$$

Solution. The matrix S and the vector $\widetilde{Y}$ are as bellow:

$$S = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \quad \widetilde{Y} = \begin{pmatrix} r \\ 1+r \\ 2r \\ 2-r \\ 3-r \\ 3-r \end{pmatrix}$$

The singular value decomposition of S is given by $S = U\Sigma V^T$, where

$$U = \begin{pmatrix} -0.3322 & 0.3586 & 0.2236 & -0.3737 & 0.4520 & -0.6059 \\ -0.3322 & 0.1195 & -0.6708 & -0.3737 & -0.5138 & -0.1475 \\ -0.5285 & 0.5976 & -0.0000 & 0.4698 & 0.0309 & 0.3767 \\ 0.3322 & 0.3586 & 0.2236 & 0.3737 & -0.5446 & -0.5242 \\ 0.3322 & 0.1195 & -0.6708 & 0.3737 & 0.4829 & -0.2292 \\ 0.5285 & 0.5976 & -0.0000 & -0.4698 & 0.0309 & 0.3767 \end{pmatrix},$$

$$\Sigma = \begin{pmatrix} 2.9618 & 0 & 0 & 0 \\ 0 & 2.6458 & 0 & 0 \\ 0 & 0 & 1.4142 & 0 \\ 0 & 0 & 0 & 0.4775 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$V = \begin{pmatrix} -0.5812 & 0.6325 & -0.3162 & 0.4028 \\ -0.4028 & 0.3162 & 0.6325 & -0.5812 \\ 0.5812 & 0.6325 & -0.3162 & -0.4028 \\ 0.4028 & 0.3162 & 0.6325 & 0.5812 \end{pmatrix}.$$

The Moore-Penrose pseudoinverse of S will be

$$\begin{aligned} S^+ &= V\Sigma^+U^T \\ &= \begin{pmatrix} -0.2143 & -0.0714 & 0.6429 & 0.2857 & 0.4286 & -0.3571 \\ 0.6429 & 0.2143 & -0.4286 & -0.3571 & -0.7857 & 0.5714 \\ 0.2857 & 0.4286 & -0.3571 & -0.2143 & -0.0714 & 0.6429 \\ -0.3571 & -0.7857 & 0.5714 & 0.6429 & 0.2143 & -0.4286 \end{pmatrix}. \end{aligned}$$

So, we obtain the solution of our system as $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2) = S^+\widetilde{Y}$, where

$$\begin{aligned} \widetilde{x}_1 &= (0.7143 + 0.6429r, 1.7143 - 0.3571r), \\ \widetilde{x}_2 &= (-1.1429 + 0.5714r, -0.1429 - 0.4286r). \end{aligned}$$

In this example $\underline{x}_1 \leq \overline{x}_1$ and $\underline{x}_2 \leq \overline{x}_2$. Moreover, $\underline{x}_1$ and $\underline{x}_2$ are non-decreasing functions, and $\overline{x}_1$, $\overline{x}_2$ are non-increasing functions over [0,1]. So, according to Definitions 3.1 and 3.3, the fuzzy solution $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2) = S^+\widetilde{Y}$ is the strong fuzzy solution. See Fig. 1. $\square$

Example 4.2. *Solve the following* 2×3 *fuzzy system of linear equations.*

$$\begin{cases} 3\widetilde{x}_1 - \widetilde{x}_2 + 2\widetilde{x}_3 &= (r, 2-r), \\ \widetilde{x}_1 + 3\widetilde{x}_2 + \widetilde{x}_3 &= (1+r, 3-r). \end{cases}$$

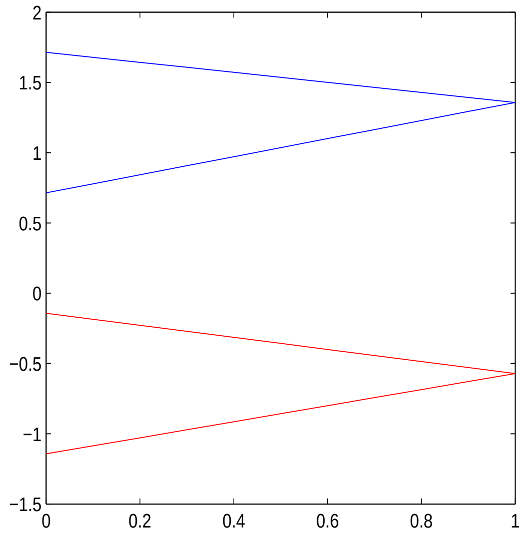

Fig. 1. The strong fuzzy solution

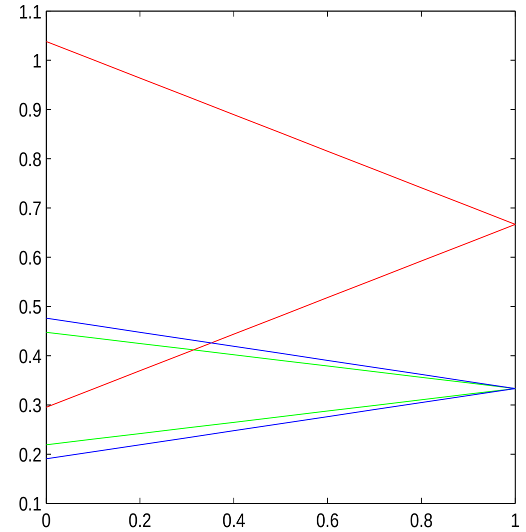

Fig. 2. The strong fuzzy solution

Solution. We have

$$S = \begin{pmatrix} 3 & 0 & 2 & 0 & -1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 & 1 \end{pmatrix}, \quad \widetilde{Y} = \begin{pmatrix} r \\ 1+r \\ 2-r \\ 3-r \end{pmatrix}.$$

So, the solution of our problem will be $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3) = S^+\widetilde{Y}$ where

$$\begin{aligned}\widetilde{x}_1 &= (0.2190 + 0.1143r, 0.4476 - 0.1143r)\\ \widetilde{x}_2 &= (0.2952 + 0.3714r, 1.0381 - 0.3714r)\\ \widetilde{x}_3 &= (0.1905 + 0.1429r, 0.4762 - 0.1429r)\end{aligned}$$

According to Definitions 3.1 and 3.3, we conclude that the fuzzy solution $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)$ is the strong fuzzy solution of problem. See Fig. 2. $\square$

Example 4.3. *Solve the following* 2×2 *fuzzy system of linear equations.*

$$\begin{cases} \widetilde{x}_1 - \widetilde{x}_2 & = & (r, 2-r) \\ \widetilde{x}_1 + 3\widetilde{x}_2 & = & (4+r, 7-2r) \end{cases}$$

Solution. We have

$$S = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 3 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}, \quad \widetilde{Y} = \begin{pmatrix} r \\ 4+r \\ 2-r \\ 7-2r \end{pmatrix}.$$

So, the solution of our problem will be $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2) = S^+\widetilde{Y}$, where

$$\begin{aligned}\widetilde{x}_1 &= (1.375 + 0.625r, 2.875 - 0.875r)\\ \widetilde{x}_2 &= (0.875 + 0.125r, 1.375 - 0.375r)\end{aligned}$$

According to Definitions 3.1 and 3.3, $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2)$ is the strong fuzzy solution. See Fig. 3. $\square$

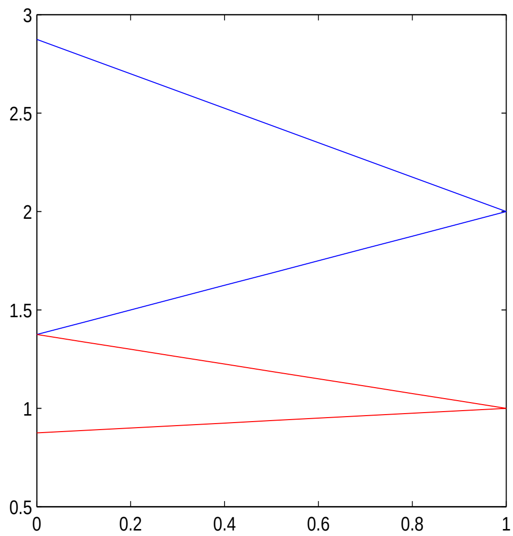

Fig. 3. The strong fuzzy solution

Example 4.4. *Solve the following* 3×2 *fuzzy system of linear equations.*

$$\begin{cases} \widetilde{x}_1 + 3\widetilde{x}_2 = (1+r, 3-r) \\ 4\widetilde{x}_1 - \widetilde{x}_2 = (r, 2-r) \\ -\widetilde{x}_1 + 3\widetilde{x}_2 = (2-2r, 1+2r) \end{cases}$$

Solution. The matrix S and the vector $\widetilde{Y}$ are as bellow:

$$S = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 4 & 0 & 0 & -1 \\ 0 & 3 & -1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & -1 & 4 & 0 \\ -1 & 0 & 0 & 3 \end{pmatrix}, \quad \widetilde{Y} = \begin{pmatrix} 1+r \\ r \\ 2-2r \\ 3-r \\ 2-r \\ 1+2r \end{pmatrix}.$$

The singular value decomposition of S is given by $S = U\Sigma V^T$, where

$$U = \begin{pmatrix} -0.3791 & -0.2364 & -0.5084 & 0.3260 & -0.6466 & 0.1320 \\ -0.4610 & 0.5060 & -0.4344 & -0.5361 & 0.1859 & 0.1437 \\ -0.3791 & -0.4336 & -0.2297 & 0.3260 & 0.7086 & -0.0841 \\ 0.3791 & -0.2364 & -0.5084 & -0.3260 & -0.0350 & -0.6591 \\ 0.4610 & 0.5060 & -0.4344 & 0.5361 & 0.1859 & 0.1437 \\ 0.3791 & -0.4336 & -0.2297 & -0.3260 & 0.0969 & 0.7070 \end{pmatrix},$$

$$\Sigma = \begin{pmatrix} 5.3397 & 0 & 0 & 0 \\ 0 & 4.7467 & 0 & 0 \\ 0 & 0 & 3.8038 & 0 \\ 0 & 0 & 0 & 2.9133 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$V = \begin{pmatrix} -0.4874 & 0.4680 & -0.5301 & -0.5123 \\ -0.5123 & -0.5301 & -0.4680 & -0.4874 \\ 0.4874 & 0.4680 & -0.5301 & 0.5123 \\ 0.5123 & -0.5301 & -0.4680 & -0.4874 \end{pmatrix}.$$

The generalized inverse of S will be

$$\begin{aligned} S^+ &= V\Sigma^+U^T \\ &= \begin{pmatrix} 0.0248 & 0.2467 & -0.0335 & 0.0703 & -0.0259 & 0.0120 \\ 0.1798 & -0.0485 & 0.1675 & -0.0020 & 0.0424 & -0.0142 \\ 0.0703 & -0.0259 & 0.0120 & 0.0248 & 0.2467 & -0.0335 \\ -0.0020 & 0.0424 & -0.0142 & 0.1798 & -0.0485 & 0.1675 \end{pmatrix}. \end{aligned}$$

So, the solution of our problem will be $\widetilde{X} = (\widetilde{x}_1, \widetilde{x}_2) = S^+\widetilde{Y}$, where

$$\begin{aligned} \widetilde{x}_1 &= (0.1288 + 0.0318r, 0.6286 - 0.3181r), \\ \widetilde{x}_2 &= (0.5796 - 0.2727r, 0.5795 + 0.2727r). \end{aligned}$$

Here, $\underline{x}_2$ is not a non-decreasing function, and $\overline{x}_2$ is not a non-increasing function over [0,1]. So, x_2 is not a fuzzy number. Therefor, the solution of our system is a weak solution as bellow:

$$\widetilde{u}_1 = (0.1288 + 0.0318r, 0.6286 - 0.3181r)$$
$$\widetilde{u}_2 = (0.3069, 0.7595 + 0.2727r)$$

□

5 Conclusions

In this work, the presented method gives a simple way to solve the fuzzy system of m linear equations with n variables. In this method, the original fuzzy system is replaced by a $2m \times 2n$ crisp linear system with coefficient matrix S. The matrix S may be singular even if the original coefficients matrix is nonsingular. In order to find the solution of our problem, we obtain the singular value decomposition of S. By using this decomposition, we calculate the Moore-Penrose pseudoinverse of S and get the solution of our problem.

Competing Interests

Author has declared that no competing interests exist.

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